CONVERGENCE OF CASCADE ALGORITHMS IN SOBOLEV SPACES FOR PERTURBED REFINEMENT MASKS

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Abstract

In this paper the convergence of the cascade algorithm in a Sobolev space is considered if the refinement mask is perturbed. It is proved that the cascade algorithm corresponding to a slightly perturbed mask converges. Moreover, the perturbation of the resulting limit function is estimated in terms of that of the masks.

Key words and phrases: cascade algorithm, Sobolev space, joint spectral radius, perturbation of refinable functions

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§1. Introduction

In this paper we are concerned with the following problem: Given a compactly supported multivariate refinable function \( \phi \), how does perturbation of its finite refinement mask affect the convergence of the cascade algorithm? Further, if the cascade algorithm for the perturbed mask also converges, how the resulting limit function is related with \( \phi \)?

We say that a compactly supported function \( \phi \) is \( M \)-refinable if it satisfies a refinement equation

\[
\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(M \cdot - \alpha),
\]

where the finitely supported sequence \( a = (a(\alpha))_{\alpha \in \mathbb{Z}^s} \) is called the refinement mask. The \( s \times s \) matrix \( M \) is called a dilation matrix. We suppose that its entries are integers and that \( \lim_{k \to \infty} M^{-k} = 0 \). Throughout the paper we assume that \( M \) is isotropic. This means that there is an invertible matrix \( \Lambda \) such that

\[
\Lambda M \Lambda^{-1} = \text{diag}(\sigma_1, \ldots, \sigma_s)
\]

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with $|\sigma_1| = \cdots = |\sigma_s| = m^{1/s} = \varrho(M)$, where $m := |\det M|$ and where $\varrho(M)$ is the spectral radius of $M$.

Let the Fourier transform $\hat{f}$ of a function $f \in L_1(\mathbb{R}^s)$ be defined by

$$\hat{f}(\omega) := \int_{\mathbb{R}^s} f(x)e^{-ix \cdot \omega} \, dx, \quad \omega \in \mathbb{R}^s,$$

where $x \cdot \omega$ denotes the inner product of two vectors $x$ and $\omega$ in $\mathbb{R}^s$. The Fourier transform is naturally extended to the space of all compactly supported distributions. We can rewrite the equation (1.1) as

$$\hat{\phi}(M^T \omega) = H_a(\omega)\hat{\phi}(\omega), \quad \omega \in \mathbb{R}^s,$$

where the refinement mask symbol

$$H_a(\omega) = \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)e^{-i\omega \cdot \alpha}, \quad \omega \in \mathbb{R}^s$$

is a (multivariate) trigonometric polynomial.

Looking at the refinement equation (1.1) as a functional equation, one can give necessary and sufficient conditions for the mask $a$ to ensure existence, uniqueness and regularity of the solution $\phi_a$ (see e.g. [1] for $M = 2I$). Provided that

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = m,$$

there exists a unique compactly supported distribution $\phi_a$ with $\hat{\phi}_a(0) = 1$ satisfying (1.1) (see e.g. [1,22]). Throughout the paper we assume that the condition (1.3) holds for the refinement masks considered.

Before posing the problem more explicitly, we need to review some notations. For $1 \leq p \leq \infty$, the norm of $L_p(\mathbb{R}^s)$ is denoted by $\| \cdot \|_p$. Let

$$W_p(\mathbb{R}^s) := \begin{cases} L_p(\mathbb{R}^s), & 1 \leq p < \infty, \\ C_u(\mathbb{R}^s), & p = \infty, \end{cases}$$

where $C_u(\mathbb{R}^s)$ is the space of uniformly continuous and bounded functions on $\mathbb{R}^s$ equipped with norm $\| \cdot \|_\infty$. Further, we use the convention $1/\infty = 0$.

Let $\mathbb{Z}_+^s$ be the set of nonnegative integers and

$$\mathbb{Z}_+^s := \left\{(\mu_1, \ldots, \mu_s) \in \mathbb{Z}^s : \mu_i \geq 0 \quad \forall i = 1, \ldots, s\right\}.$$ 

For any multi-integer $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{Z}_+^s$, let $|\mu| := \mu_1 + \cdots + \mu_s$, $\mu! := \mu_1! \cdots \mu_s!$, and $x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s}$. Further, $

\Pi_n$
denotes the linear span of \( \{ x^\nu : |\nu| \leq n \} \). For two multi-integers \( \mu = (\mu_1, \ldots, \mu_s) \) and \( \nu = (\nu_1, \ldots, \nu_s) \), we say \( \nu \leq \mu \) if \( \nu_i \leq \mu_i \) for all \( i = 1, \ldots, s \). For \( \nu \leq \mu \), we use \( (\nu)^\mu \) to denote \( \frac{\mu!}{(\nu!)(\mu-\nu)!} \).

For \( n \in \mathbb{Z}_+ \), the Sobolev space \( W^n_p(\mathbb{R}^s) \) is the set of all tempered distributions \( f \) such that \( D^\mu f \in W_p(\mathbb{R}^s) \) for \( |\mu| \leq n \), where \( D^\mu = D_1^{\mu_1} \cdots D_s^{\mu_s} \) and \( D_j := \frac{\partial}{\partial x_j} \) \( (j = 1, \ldots, s) \) denote the partial derivatives. Clearly, \( W^n_p(\mathbb{R}^s) \) is a Banach space with the norm

\[
||f||_{W^n_p(\mathbb{R}^s)} := \sum_{|\mu| \leq n} ||D^\mu f||_p, \quad 1 \leq p \leq \infty.
\]

Let \( E \) be a complete set of representatives of distinct cosets of the quotient group \( \mathbb{Z}^s/M \mathbb{Z}^s \). Thus, each element \( \alpha \in \mathbb{Z}^s \) can be uniquely represented as \( \alpha = \alpha + M \gamma, \gamma \in E \) and \( \gamma \in \mathbb{Z}^s \). It is known that the cardinality of \( E \) is equal to \( m = |\det M| \). Without loss of generality, we can assume that \( 0 \in E \).

Denote by \( \ell(p)(\mathbb{Z}^s) \) the space of all complex-valued sequences. Let \( \ell(p)(\mathbb{Z}^s) \) be the space of complex-valued sequences \( \lambda = (\lambda(\alpha))_{\alpha \in \mathbb{Z}^s} \) such that \( ||\lambda||_p < \infty \), where

\[
||\lambda||_p := \left\{
\begin{array}{ll}
\left( \sum_{\alpha \in \mathbb{Z}^s} |\lambda(\alpha)|^p \right)^{1/p}, & 1 \leq p < \infty, \\
\sup_{\alpha \in \mathbb{Z}^s} |\lambda(\alpha)|, & p = \infty.
\end{array}
\right.
\]

(Observe that the norms for \( W_p(\mathbb{R}^s) \) and \( \ell_p(\mathbb{Z}^s) \) both are abbreviated by \( \| \cdot \|_p \), the particular interpretation will always follow from the context.)

Denote by \( \ell_0(\mathbb{Z}^s) \) the space of sequences of finite support. For \( \lambda \in \ell_0(\mathbb{Z}^s) \) let \( \text{supp} \lambda := \{ \alpha \in \mathbb{Z}^s : \lambda(\alpha) \neq 0 \} \).

Given a compactly supported initial function \( \phi_0 \in L_p(\mathbb{R}^s) \) we define a sequence \( (\phi_k)_{k \geq 1} \) by iteration \( \phi_k := Q_\alpha \phi_{k-1}, k = 1, 2, \ldots, \) where \( Q_\alpha : L_p(\mathbb{R}^s) \to L_p(\mathbb{R}^s) \) is the cascade operator associated with the finite mask \( a \),

\[
Q_\alpha f := \sum_{\beta \in \mathbb{Z}^s} a(\beta) f(M \cdot \beta).
\]

We say that the cascade algorithm converges for \( \phi_0 \) in \( W^n_p(\mathbb{R}^s) \)-norm \( (1 \leq p \leq \infty) \) if the sequence \( (Q^k_\alpha \phi_0)_{k \geq 1} \) converges in \( W^n_p(\mathbb{R}^s) \)-norm. In this case, it has been proved in \([3]\) that \( \phi_0 \) is necessarily contained in the space

\[
W_n := \{ f \in W^n_p(\mathbb{R}^s) \text{ compactly supp. : } D^\mu \hat{f}(2 \pi \alpha) = 0 \ \forall \alpha \in \mathbb{Z}^s \setminus \{0\}, |\mu| \leq n \}
\]

(1.5) The cascade operator \( Q_\alpha \) is closely connected with the subdivision operator \( S_\alpha : \ell_0(\mathbb{Z}^s) \to \ell_0(\mathbb{Z}^s) \) associated with the mask \( a \),

\[
S_\alpha v(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M \beta) v(\beta), \quad \alpha \in \mathbb{Z}^s.
\]
Denoting $a_k := S^k_a \delta$, where $\delta$ is the impulse sequence given by $\delta(a) = 0$ for $a \in \mathbb{Z}^* \setminus \{0\}$ and $\delta(0) = 1$, we have $a_1 = a$ and
\[
a_k(a) = \sum_{\beta \in \mathbb{Z}^*} a_{k-1}(\beta) a(a - M\beta), \quad a \in \mathbb{Z}^*, \quad k \geq 2. \tag{1.6}
\]
It can easily be verified by induction (see [11]) that for $f \in L_p(\mathbb{R}^*)$
\[
Q^k f = \sum_{\alpha \in \mathbb{Z}^*} a_k(\alpha) f(M^k \cdot -\alpha), \quad k = 1, 2, \ldots. \tag{1.7}
\]
The cascade algorithm plays an important role in computer graphics and wavelet analysis. The convergence of the cascade algorithm has been studied by many authors. Cavaretta, Dahmen and Micchelli [1] already found necessary and sufficient conditions ensuring that the subdivision scheme related to a finitely supported refinement mask with dilation matrix $M = 2I$ uniformly converges to a continuous limit function. In the $L_2$-norm the convergence of the cascade algorithm has been shown by Strang [28] in the univariate case, by Lawton, Lee and Shen [23] in the multivariate case and by Shen [27] in the general multivariate vector case. Jia [15] considered the convergence of subdivision schemes in the univariate setting for general $L_p$-spaces; the multivariate $L_p$-case is completely settled in Han and Jia [11]. For the univariate vector case we refer to Jia, Riemenschneider and Zhou [21] and to Micchelli and Sauer [24,25]. Convergence in $W^n_p(\mathbb{R}^*)$ has firstly been discussed by Jia, Jiang and Lee [19]. For scalar subdivision schemes in Sobolev spaces we also refer to Goodman and Lee and to Micchelli and Sauer [6,26]. The cascade algorithm in Besov spaces has been considered by Sun [29], Chen, Jia and Riemenschneider [3] and Zhou [31] have studied this problem in $W^n_p(\mathbb{R}^*)$ for $1 \leq p \leq \infty$.

In practice, one often has to handle perturbed refinement masks. In fact, coefficients are generally irrational or rational numbers which need to be truncated in floating point arithmetics. Heil and Collela [12] were the first, who studied how such truncation affects the refinable function in the univariate $L_\infty$-case (see also [13]). Further discussions on the effect of perturbed scaling coefficients in the univariate case can be found in Villemoes [30] and in Daubechies and Huang [4]. Villemoes even showed that, under certain conditions, membership of a refinable function in a Besov class is stable under perturbations.

More recently, Han [7,8] provided a sharp error estimate for multivariate refinable functions in any $L_p$-norm. His idea has been adopted to perturbed matrix masks in the univariate $L_p$-case by Han and Hogan (see [10]).

In particular, Han could show the following result in [7,8]: If the cascade algorithm related to a mask $a$ converges for $\phi_0$ in $L_p$-norm, and if $b$ is an only slightly perturbed mask, i.e., $\|a - b\|_1 < \eta$ for a sufficiently small $\eta > 0$, and $b$ satisfies the sum rules of order 1 (see Section 2 for the notion of sum rules of order $n$), then the cascade algorithm associated with $b$ also converges for $\phi_0$ in $L_p$-norm and we have
\[
\|Q^k_b \phi_0 - Q^k_a \phi_0\|_p \leq C \|a - b\|_1, \quad k \geq 1. \tag{1.8}
\]
Here the constant $C$ depends on the refinement mask $a$ under consideration as well as on $p, 1 \leq p \leq \infty$. However, it is independent of the perturbed mask $b$ and of $k$.

In this paper we want to generalize the above result to cascade algorithms converging in Sobolev spaces.

Compared with the $L_p$-case, we have to overcome some difficulties due to the handling with function derivatives requiring another approach. In fact the proof of the main result is based on two new key ingredients.

The first basic idea to obtain the wanted estimate is the observation, that for some suitable initial function $\phi_0$ the following inequality holds: There exists a positive constant $c$ with

$$
\sum_{|\mu|=n} \| D^\mu Q_a^k \phi_0 \|_p \leq c \ m^{(n/s-1/p)k} \sum_{|\mu|=n} \| \Delta^\mu a_k \|_p
$$

for all $k = 1, 2, \ldots$ (see Theorem 3.2). Here $\Delta^\mu$ denotes the $\mu$th difference operator (see Section 2) and $a_k$ is the iterated subdivision operator applied to $\delta$ in (1.6).

The second key ingredient for the wanted estimate is the inequality

$$
\| \Delta^\mu a_k - \Delta^\mu b_k \|_p \leq c \| a - b \|_1 \ m^{(-n/s+1/p)k} \quad \forall |\mu| = n, \ k = 1, 2, \ldots
$$

(see Lemma 4.3). The proof of this inequality requires exact analysis of the connection between convergence and boundedness of $(Q_a^k \phi_0)_{k \geq 0}$ (resp. $(Q_b^k \phi_0)_{k \geq 0}$) and the behavior of $\| \Delta^\mu a_k \|_p$ (resp. $\| \Delta^\mu b_k \|_p$) with $|\mu| = n$, even slightly extending the known results on convergence of cascade algorithms in Sobolev spaces (see [3]).

In Section 2, we recall some important definitions and results from [3,11]. In particular, two equivalent characterizations of the convergence of cascade algorithms in $W^m_p(\mathbb{R}^s)$ are given in terms of the joint spectral radius and of the subdivision operator. In Section 3 we construct a special initial function satisfying the above useful inequality. Further, an implicit relation between the boundedness and convergence of a cascade algorithm in different Sobolev spaces is established. Section 4 is devoted to the generalization of (1.8) to Sobolev spaces.

§2. JOINT SPECTRAL RADIUS

In the study of convergence of the cascade algorithm, the joint spectral radius of linear operators is a useful tool. The uniform joint spectral radius was employed in [5] to investigate the regularity of refinable functions. For $1 \leq p < \infty$, the $p$-joint spectral radius was introduced and applied to the study of $L_p$-convergence of cascade algorithms by Jia [15]. We cite from [15] the definition of $p$-norm joint spectral radius for the convenience of the reader.

Let $V$ be a finite-dimensional space with norm $|| \cdot ||$. For a linear operator $A$ on $V$ define

$$
|||A||| := \max \{ ||Av|| : ||v|| = 1 \}.
$$

Let $A$ be a finite collection of linear operators on a finite-dimensional vector space $V$. For a positive integer $k$ we denote by $A^k$ the Cartesian power of $A$:

$$
A^k = \{(A_1, \ldots, A_k) : A_1, \ldots, A_k \in A \}.
$$
Now let
\[ \|A^k\|_p := \begin{cases} \left( \sum_{(A_1, \ldots, A_k) \in \mathcal{A}^k} \|A_1 \cdots A_k\|_p^p \right)^{1/p}, & 1 \leq p < \infty, \\ \max \{ \|A_1 \cdots A_k\| : (A_1, \ldots, A_k) \in \mathcal{A}^k \}, & p = \infty. \end{cases} \]

The \( p \)-norm joint spectral radius of \( \mathcal{A} \) is defined to be
\[ \rho_p(\mathcal{A}) := \lim_{k \to \infty} \|A^k\|_p^{1/k}. \]  
This limit indeed exists and does not depend on the choice of norm on \( V \). Moreover, we have
\[ \lim_{k \to \infty} \|A^k\|_p^{1/k} = \inf_{k \geq 1} \|A^k\|_p^{1/k}. \]  

Further, let for \( v \in V \)
\[ \|A^k v\|_p := \begin{cases} \left( \sum_{(A_1, \ldots, A_k) \in \mathcal{A}^k} \|A_1 \cdots A_k v\|_p^p \right)^{1/p}, & 1 \leq p < \infty, \\ \max \{ \|A_1 \cdots A_k v\| : (A_1, \ldots, A_k) \in \mathcal{A}^k \}, & p = \infty. \end{cases} \]

Let us come back to our problem. For a finite refinement mask \( a \), we consider \( m \) operators \( A_\varepsilon, \varepsilon \in E \), on \( \ell_0(\mathbb{Z}^k) \) defined by the biinfinite matrices
\[ A_\varepsilon(\alpha, \beta) = a(\varepsilon + M\alpha - \beta), \quad \alpha, \beta \in \mathbb{Z}^k. \]  

Hence
\[ A_\varepsilon v(\alpha) = \sum_{\beta \in \mathbb{Z}^k} a(\varepsilon + M\alpha - \beta) v(\beta), \quad v \in \ell_0(\mathbb{Z}^k). \]

Let now \( \mathcal{A} \) be the finite collection of \( A_\varepsilon, \varepsilon \in E \). There is a simple relation between \( a_k \) in (1.6) and the matrices \( A_\varepsilon, \varepsilon \in E \) in (2.3). Let \( a \in \mathbb{Z}^k \) and \( k \) be a positive number. Then there are (uniquely defined) \( \varepsilon_1, \ldots, \varepsilon_k \in E \) and \( \gamma \in \mathbb{Z}^k \) such that
\[ \alpha = \varepsilon_1 + M\varepsilon_2 + \cdots + M^{k-1}\varepsilon_k + M^k\gamma \]  

and we have (see [11], Lemma 2.1)
\[ a_k(\alpha - \beta) = A_{\varepsilon_k} \cdots A_{\varepsilon_1}(\gamma, \beta) \quad \forall \beta \in \mathbb{Z}^k. \]  

For two sequences \( u \in \ell_p(\mathbb{Z}^k) \) and \( v \in \ell_0(\mathbb{Z}^k) \), the discrete convolution \( u * v \in \ell_p(\mathbb{Z}^k) \) is defined by
\[ (u * v)(\alpha) = \sum_{\beta \in \mathbb{Z}^k} u(\alpha - \beta)v(\beta), \quad \alpha \in \mathbb{Z}^k. \]

It follows from equality (2.5) that, for any \( v \in \ell_0(\mathbb{Z}^k) \),
\[ (a_k * v)(\alpha) = A_{\varepsilon_k} \cdots A_{\varepsilon_1} v(\gamma), \]
with \( a = \varepsilon_1 + M\varepsilon_2 + \ldots + M^{k-1}\varepsilon_k + M^k \gamma \) and consequently for \( 1 \leq p < \infty \),

\[
\| a_k \cdot v \|_p^p = \sum_{\varepsilon_1, \ldots, \varepsilon_k \in E} \| A_{\varepsilon_k} \cdots A_{\varepsilon_1} v \|_p^p = \| A^k v \|_p^p.
\]

Let \( e_j \) be the \( j \)th coordinate unit vector of \( \mathbb{R}^s \), \( j = 1, 2, \ldots, s \). Recall that, for any \( j = 1, 2, \ldots, s \) and a function \( f \) defined on \( \mathbb{R}^s \) the difference operator \( \Delta_j \) is given by

\[
\Delta_j f := f(\cdot) - f(\cdot - e_j).
\]

Analogously, let the difference operator \( \Delta_j \) be defined for sequences \( \lambda \in \ell(\mathbb{Z}^s) \), by \( \Delta_j \lambda = \lambda(\cdot) - \lambda(\cdot - e_j) \). Further, for any \( \mu = (\mu_1, \ldots, \mu_s) \in \mathbb{Z}^s \), denote \( \Delta^\mu = \Delta_{\mu_1} \cdots \Delta_{\mu_s} \) by \( \Delta^\mu \).

In order to give a characterization for convergence of the cascade algorithm in \( W_p^n(\mathbb{R}^s) \)-norm we introduce the subspace

\[
V_n := \left\{ v = (v(\alpha))_{\alpha \in \mathbb{Z}^s} \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha \in \mathbb{Z}^s} \alpha^\mu v(\alpha) = 0, |\mu| \leq n \right\}. \tag{2.7}
\]

Observe that \( V_n = \text{span} \{ \Delta^\mu(\cdot - \beta) : \beta \in \mathbb{Z}^s, |\mu| = n+1 \} \), where \( \delta \) is the impulse sequence. Furthermore, one can construct a finite set \( K \subseteq \mathbb{Z}^s \) such that \( \ell(K) \) is a finite subspace of \( \ell_0(\mathbb{Z}^s) \) consisting of all sequences with support on \( K \) with the following properties:

1. \( \ell(K) \) is an invariant subspace under \( A_e \) for any \( e \in E \);
2. \( \ell(K) \) contains \( \Delta^\mu \), \( |\mu| = n+1 \).

To this end, let \( \text{supp} \ a := \{ \alpha : a(\alpha) \neq 0 \} \) and \( \Omega \) be a finite set of \( \mathbb{Z}^s \) such that \( \text{supp} \ a \cup \{0\} \subseteq \Omega \). Put \( H := \Omega - E + M\mathbb{Z}^s_{n+1} \), where \( \mathbb{Z}^s_{n+1} := \{ (\mu_1, \ldots, \mu_s) \in \mathbb{Z}^s, 0 \leq \mu_i \leq n+1, 1 \leq i \leq s \} \). (Here, the set \( \mathcal{A} + \mathcal{B} \) (or \( \mathcal{A} - \mathcal{B} \)) consists of all points \( x + y \) (or \( x - y \)) with \( x \in \mathcal{A} \) and \( y \in \mathcal{B} \).) Now let

\[
K := \mathbb{Z}^s \cap \sum_{k=1}^\infty M^{-k} H. \tag{2.8}
\]

In particular, we have \( M^{-1}(K + \Omega - E) \cap \mathbb{Z}^s \subseteq K \). It is not difficult to see that \( \ell(K) \) is invariant under \( A_e, e \in E \), i.e. for \( v \in \ell(K) \) we have \( A_e v \in \ell(K) \) (see [11], Lemma 2.3).

Then in [3] the following has been shown

**Result 2.1.** ([3]) Let \( a \in \ell_0(\mathbb{Z}^s) \) satisfy (1.3) and let \( W_n \) be given in (1.5). The cascade algorithm associated with a converges for all functions \( \phi \) in \( W_n \) in \( W_p^n(\mathbb{R}^s) \)-norm \( (1 \leq p \leq \infty) \) if and only if the following conditions are satisfied:

1. \( V_n \) is invariant under \( A_e \forall e \in E \), i.e. for \( v \in V_n \) it follows that \( A_e v \in V_n \);
2. \( \rho_p \left( \{ A_e |_{V_n \cap \ell(K)} : e \in E \} \right) < m^{-n/s+1/p} \), where \( K \) is given in (2.8).
Remarks. 1. The condition (1) in Result 2.1 is a necessary condition on the mask \( a \), it needs to be satisfied if the limit function of cascade algorithm is wanted to be in \( W_p^n(\mathbb{R}^s) \). Moreover, (1) is equivalent with the sum rules of order \( n + 1 \), saying that for any \( p \in \Pi_n \)

\[
\sum_{\alpha \in \mathbb{Z}^s} p(M\alpha + \varepsilon) a(M\alpha + \varepsilon) = \sum_{\alpha \in \mathbb{Z}^s} p(M\alpha) a(M\alpha) \quad \forall \varepsilon \in E. \tag{2.9}
\]

This equivalence has already been shown in [18], Theorem 5.2 (see also [14], Theorem 3.4.12.). We want to remark that the condition (1), or equivalently, the sum rules of order \( n + 1 \) are also necessary for reproduction of polynomials up to total degree \( n \) in the shift-invariant space \( \hat{S}(\phi) \) generated by the integer translates of the \( M \)-refinable function \( \phi \) (see [17]).

2. The condition (2) in Result 2.1 can be seen as a generalization of the result in [11], where the convergence of cascade algorithms in \( L_p \)-spaces is shown.

Since \( K \) is a finite set, the \( p \)-norm joint spectral radius needs to be determined only in the finite dimensional space \( V_n \cap \ell(K) \).

The following lemma justifies the definition of the set \( K \) in (2.8). Here we consider the action of operators \( A_{\varepsilon}, \varepsilon \in E \), on the sequences with supports contained in any fixed finite set \( K_1 \subseteq \mathbb{Z}^s \).

**Lemma 2.2.** Let \( K \) be defined by (2.8). Then for any finite set \( K_1 \subseteq \mathbb{Z}^s \), there is a positive integer \( j \) such that

\[
A_{\varepsilon_j} \cdots A_{\varepsilon_1} v \in \ell(K) \quad \forall v \in \ell(K_1) \text{ and } \varepsilon_1, \ldots, \varepsilon_j \in E. \tag{2.10}
\]

Consequently, for any integer \( k > j \)

\[
A_{\varepsilon_k} \cdots A_{\varepsilon_1} v \in \ell(K) \quad \forall v \in \ell(K_1) \text{ and } \varepsilon_1, \ldots, \varepsilon_k \in E. \tag{2.11}
\]

**Proof.** For any \( v \in \ell(K_1) \) we have \( \operatorname{supp} A_{\varepsilon} v \subseteq M^{-1}(K_1 + \Omega - E) \cap \mathbb{Z}^s, \varepsilon \in E \). Iterative application yields for any integer \( j > 0 \)

\[
\operatorname{supp} A_{\varepsilon_j} \cdots A_{\varepsilon_1} v \subseteq \left( M^{-j}(K_1 + \Omega - E) \cap \mathbb{Z}^s \right) + \left( M^{-j+1}(\Omega - E) \cap \mathbb{Z}^s \right) + \cdots + \left( M^{-1}(\Omega - E) \cap \mathbb{Z}^s \right),
\]

where \( \Omega \) contains the support of \( a \).

Since \( M \) is isotropic, there is a constant \( c \) being independent of \( j \) such that

\[
\| M^{-j}\omega \| \leq cm^{-j/2}\| \omega \| \quad \forall \omega \in \mathbb{R}^s \text{ and } j = 1, 2, \ldots. \tag{2.12}
\]

with \( m = |\det M| > 1 \) (see e.g. [17], Lemma 6.1). Therefore we can find an integer \( j \) such that such that, for all \( \alpha \in K_1 + \Omega - E \), \( M^{-j}\alpha \in (-1,1)^s \), i.e. \( M^{-j}(K_1 + \Omega - E) \cap \mathbb{Z}^s \in \{0,\{0\}\} \) and (2.10) holds. Since \( \ell(K) \) is an invariant subspace under \( A_{\varepsilon} \) for any \( \varepsilon \in E \), (2.11) follows for any \( k > j \).

There is a second way to characterize the convergence of the cascade algorithm using the subdivision operator \( S_a \).
Theorem 2.3. Let $a \in \ell_q(\mathbb{Z}^s)$ satisfy (1.3). Then the cascade algorithm associated with a converges for all functions in $W_n$ in $W_p^s(\mathbb{R}^s)$-norm ($1 \leq p \leq \infty$) if and only if

$$
\lim_{k \to \infty} m^{k(n/s-1/p)} \| \Delta^\mu a_k \|_p = 0 \quad \forall |\mu| = n + 1,
$$

(2.13)

where $a_k = S(a, \delta)$.

The proof of this theorem will be given in the next section.

§3. Differential and Difference Operator

We now turn our attention to the norms $\| Q_k^j \phi_0 \|_{W^p_0}$. The goal is to estimate them in terms of sequence norms deduced from $a_k$. In particular, we shall show in this section, that boundedness of $(Q_k^j \phi_{0,k})_{k \geq 1}$ (where $\phi_{0,k}$ is a suitably chosen initial function in $W_n$) implies convergence of the cascade algorithm on $W_{n-1}$ in $W_p^{n-1}(\mathbb{R}^s)$-norm.

Let $f$ be a differentiable function on $\mathbb{R}^s$ and let $\mathcal{D} := (D_1, \ldots, D_s)^T$ with $D_j = \frac{\partial}{\partial x_j}$. Then the chain rule for differentiation gives

$$
\mathcal{D}(f(M^j \cdot))(x) = (M^T)^j \mathcal{D} f(M^j x), \quad x \in \mathbb{R}^s,
$$

where $M^T$ is the transpose of $M$. Since $M$ is isotropic, there exists an invertible matrix $\Lambda$ such that $\Lambda M^T \Lambda^{-1} = \text{diag} (\sigma_1, \ldots, \sigma_s)$. Hence we have

$$
\Lambda \mathcal{D}(f(M^j \cdot))(x) = \text{diag} (\sigma_1^k, \ldots, \sigma_s^k) \Lambda \mathcal{D} f(M^j x).
$$

Let $q_j(\mathcal{D}) := \Lambda_j \mathcal{D}$, where $\Lambda_j$ denotes the $j$th row of $\Lambda$ ($j = 1, \ldots, s$), and for any $\mu = (\mu_1, \ldots, \mu_s)^T \in \mathbb{Z}^s_+$, let $q_\mu(\mathcal{D}) := q_1(\mathcal{D})^{\mu_1} \cdots q_s(\mathcal{D})^{\mu_s}$. Considering the last equation componentwisely, we have for any $f \in W^s_p(\mathbb{R}^s)$

$$
q_j(\mathcal{D})(f(M^j \cdot))(x) = \sigma_j^k (q_j(\mathcal{D}) f)(M^j x), \quad j = 1, \ldots, s,
$$

and hence

$$
q_\mu(\mathcal{D})(f(M^j \cdot))(x) = (\sigma_1^{\mu_1 \cdot} \cdots \sigma_s^{\mu_s \cdot}) (q_\mu(\mathcal{D}) f)(M^j x), \quad x \in \mathbb{R}^s
$$

(3.1)

(see also [19,22,31]).

It is easily seen that the operators $q_\mu(\mathcal{D})$ may be expressed as

$$
q_\mu(\mathcal{D}) = \sum_{|\nu| = |\mu|} c_{\mu, \nu} D^\nu,
$$

where $c_{\mu, \nu}$ are determined by $\Lambda$ and $D^\nu = D_1^{\nu_1} \cdots D_s^{\nu_s}$. Since $\Lambda$ is invertible, there exists a positive number $\kappa$ satisfying, for any $f \in W^s_p(\mathbb{R}^s)$,

$$
\kappa^{-1} \sum_{|\nu| = n} |(D^\nu f)(x)| \leq \sum_{|\mu| = n} |q_\mu(D) f(x)| \leq \kappa \sum_{|\mu| = n} |(D^\nu f)(x)|, \quad x \in \mathbb{R}^s.
$$
Applying this equivalence and (3.1), we find for any \( f \in W^n_p(\mathbb{R}^s) \)

\[
\kappa^{-1} m^{s_{k/s}} \sum_{|\nu|=n} |(D^\nu f)(M^k x)| \leq \sum_{|\nu|=n} |D^\nu(f(M^k))(x)| \\
\leq \kappa m^{s_{k/s}} \sum_{|\nu|=n} |(D^\nu f)(M^k x)|, \quad x \in \mathbb{R}^s \text{ and } k = 1, 2, \ldots,
\]

where we have used that \(|\sigma_1| = \cdots = |\sigma_s| = m^{1/s}\). The second inequality has been also proved in [17]. Hence we obtain:

**Lemma 3.1.** There is a positive number \( c \) such that for any nontrivial \( f \in W^n_p(\mathbb{R}^s) \)

\[
c^{-1} m^{(s-1/p)k} \leq \sum_{|\nu|=n} \|D^\nu f(M^k x)\|_p \leq c m^{(s-1/p)k}, \quad k = 1, 2, \ldots.
\]

In these inequalities, the factor \( m^{-k/p} \) is due to the change of variables \( M^k x \rightarrow x \) in the norms.

For our considerations, we want to use a special initial function \( \phi_0 \) which is a tensor product of univariate B-splines. For \( k \in \mathbb{Z}_+ \), let \( N_k \) be the univariate forward B-spline of degree \( k \) with the knots \( 0, 1, \ldots, k+1 \), recursively given by

\[
N_k = N_{k-1} * N_0 = \int_0^1 N_{k-1}(\cdot - t) dt, \quad t \in \mathbb{R},
\]

where \( N_0 := \chi_{[0,1]} \) is the characteristic function of \([0,1]\). Furthermore, for \( \nu = (\nu_1, \ldots, \nu_s) \in \mathbb{Z}_+^s \), let \( N_\nu(x) := N_{\nu_1}(x_1) \cdots N_{\nu_s}(x_s) \), where \( x = (x_1, \ldots, x_s)^T \in \mathbb{R}^s \).

Observe that for any pair of \( \mu \) and \( \nu \in \mathbb{Z}_+^s \) with \( \mu \leq \nu \)

\[
D^\mu N_\nu = \Delta^\mu N_{\nu-\mu}.
\]  

(3.2)

A second important property of \( N_\nu \) in this context is the stability of its shifts. This means that, for any \( \nu \in \mathbb{Z}_+^s \), there is a positive number \( \kappa \), which is independent of \( \lambda \), satisfying

\[
\kappa^{-1} \||\lambda\||_p \leq \sum_{\alpha \in \mathbb{Z}^s} \|\lambda(\alpha) N_\nu(\cdot - \alpha)\|_p \leq \kappa \||\lambda\||_p \quad \forall \lambda \in \ell_p(\mathbb{Z}^s). \quad (3.3)
\]

The functions \( N_\nu \) are appropriate candidates for the initial function in the cascade algorithm. In fact,

\[
\phi_{0,n} = N_{(n+1, \ldots, n+1)} \quad (3.4)
\]

is in \( W_n \) for any \( 1 \leq p \leq \infty \) (with \( W_n \) in (1.5)).
Theorem 3.2. Let $\lambda \in \ell_0(\mathbb{Z}^s)$ and let $g$ be associated with $\lambda$ by 
\[
g = \sum_{\alpha \in \mathbb{Z}^s} \lambda(\alpha) \phi_{0,n}(M^k \cdot -\alpha).
\]
Then there exists a constant $\kappa > 0$ which is independent of $\lambda \in \ell_0(\mathbb{Z}^s)$ and $k \in \mathbb{Z}_+$, such that
\[
\kappa^{-1} m^{(n/s-1/p)} k \leq \frac{\sum_{|\mu| = n} ||D^\mu g||_p}{\sum_{|\mu| = n} ||\Delta^\mu \lambda||_p} \leq \kappa m^{(n/s-1/p)} k, \quad k = 1, 2, \ldots.
\]
(3.5)

In particular, if the sequence $(Q^k \phi_{0,n})_{k \geq 1}$ is bounded in $W_p^n(\mathbb{R}^s)$, then there is a constant $c$ being independent of $k$ such that
\[
m^{(n/s-1/p)} ||\Delta^\mu a_k||_p \leq c \quad \forall |\mu| = n, k = 1, 2, \ldots.
\]
(3.6)

Further, if the sequence $(Q^k \phi_{0,n})_{k \geq 1}$ converges in $W_p^n(\mathbb{R}^s)$ then (2.13) holds.

Proof. For $\lambda = a_k = S^k \delta$, the function $g$ associated with $\lambda$ equals to $Q^k \phi_{0,n}$ by (1.7). If the sequence $(Q^k \phi_{0,n})_{k \geq 1}$ is bounded in $W_p^n(\mathbb{R}^s)$-norm, then (3.6) follows from the first inequality in (3.5). If the sequence $(Q^k \phi_{0,n})_{k \geq 1}$ converges in $W_p^n(\mathbb{R}^s)$, then there exists a compactly supported limit function $\phi \in W_p^n(\mathbb{R}^s)$ such that
\[
||Q^k \phi_{0,n} - \phi||_{W_p^n(\mathbb{R}^s)} \to 0 \quad \text{for} \quad k \to \infty.
\]
Further, from
\[
||Q^k \phi_{0,n} - Q^k \phi_{0,n}(\cdot - M^{-k} e_j)||_{W_p^n(\mathbb{R}^s)} \\
\leq ||\phi - \phi(\cdot - M^{-k} e_j)||_{W_p^n(\mathbb{R}^s)} + 2 ||\phi - Q^k \phi_{0,n}||_{W_p^n(\mathbb{R}^s)}
\]
for all unit vectors $e_j, j = 1, \ldots, s$ we obtain that
\[
\sum_{|\mu| = n} ||\Delta^\mu D^\mu Q^k \phi_{0,n}||_p \to 0 \quad \text{for} \quad k \to \infty, j = 1, \ldots, s.
\]

Now again, for $\lambda = a_k$ we have $g = Q^k \phi_{0,n}$ and (2.13) follows from the first inequality of (3.5) as before.

Let’s now prove (3.5). Putting $f = g(M^{-k} \cdot)$ we obtain by (3.2)
\[
D^\mu f = \sum_{\alpha \in \mathbb{Z}^s} \lambda(\alpha) \Delta^\mu N_p(\cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} \Delta^\mu \lambda(\alpha) N_p(\cdot - \alpha),
\]
where $\nu = (n + 1 - \mu_1, \ldots, n + 1 - \mu_s)$. Consequently,
\[
D^\mu g = (D^\mu f)(M^k \cdot) = \sum_{\alpha \in \mathbb{Z}^s} \Delta^\mu \lambda(\alpha) N_p(M^k \cdot - \alpha).
\]
Therefore the inequalities in (3.5) are true by Lemma 3.1 and the stability property (3.3) of $N_p$. □

Remark. The necessity of (2.13) for the convergence of the cascade algorithm in $W_p^n(\mathbb{R}^s)$ has also been shown in [3], Lemma 4.3.

Now we are able to show the following relation.
Lemma 3.3. Assume that (2.13) is true for a given refinement mask $a$. Then $V_n$ in (2.7) is an invariant subspace under $A_z$ for all $\varepsilon \in E$.

Proof. For $\varepsilon \in E$ and $\mu \in \mathbb{Z}_+^*$, we define a polynomial $p_{\varepsilon, \mu} \in \Pi_{|\mu|}$ by

$$p_{\varepsilon, \mu}(x) = \sum_{\beta \in \mathbb{Z}^*} a(M\beta + \varepsilon)(M^{-1}(x - \varepsilon) - \beta)\mu.$$ 

The space $V_n$ is invariant under $A_z$ for all $\varepsilon \in E$ if and only if the mask $a$ satisfies the sum rules of order $n + 1$ in (2.9). Hence we have to show

$$p_{\varepsilon_1, \mu} = p_{\varepsilon_2, \mu} \quad \forall \varepsilon_1, \varepsilon_2 \in E \text{ and } \forall \mu \text{ with } |\mu| \leq n. \quad (3.7)$$

For $|\mu| = 0$, (3.7) has been proved in [11], Theorem 3.1. We shall prove (3.7) by induction on $n_0$ with $0 \leq n_0 \leq n$. Assume that (3.7) holds for $n_0 < n$. If it is not true for $n_0 + 1$, then there are $\varepsilon_1, \varepsilon_2 \in E$ and $\mu \in \mathbb{Z}_+^*$ with $|\mu| = n_0 + 1$ such that $p_{\varepsilon_1, \mu} \neq p_{\varepsilon_2, \mu}$. We shall show that this contradicts (2.13).

For any $\mu \in \mathbb{Z}_+^*$ and $k = 1, 2, \ldots$, let $h_{k, \mu} \in \ell_0(\mathbb{Z}^*)$ be defined by

$$h_{k, \mu}(\alpha) = \sum_{\beta \in \mathbb{Z}^*} a_k(\alpha - M\beta)\beta^\nu, \quad \alpha \in \mathbb{Z}^*,$$

where $a_k$ are given in (1.6). Observe that for any $\varepsilon \in E$

$$h_{1, \mu}(M\alpha + \varepsilon) = p_{\varepsilon, \mu}(M\alpha + \varepsilon) \quad \forall \alpha \in \mathbb{Z}^*.$$

Thus, the induction assumption (3.7) for $n_0$ implies that

$$h_{1, \mu}(\alpha) = p_{\varepsilon, \mu}(\alpha) \quad \forall \mu, |\mu| \leq n_0, \forall \varepsilon \in E \text{ and } \alpha \in \mathbb{Z}^*.$$

Consequently, since $p_{\varepsilon, \mu} \in \Pi_{|\mu|}$ we have $\Delta^\gamma h_{1, \mu} = \Delta^\gamma p_{\varepsilon, \mu} = 0$ for $|\gamma| = |\mu| + 1$ and $|\mu| \leq n_0$, i.e. $h_{1, \mu} (|\mu| \leq n_0)$ is a polynomial sequence of degree $|\mu|$.

Let now $|\mu| = n_0 + 1$. Since by assumption $p_{\varepsilon_1, \mu} \neq p_{\varepsilon_2, \mu}$ for some $\varepsilon_1, \varepsilon_2 \in E$ and some $|\mu| = n_0 + 1$ we have $h_{1, \mu}(M\alpha + \varepsilon_1) = p_{\varepsilon_1, \mu}(M\alpha + \varepsilon_1) \forall \alpha \in \mathbb{Z}^*$ but $h_{1, \mu}(M\alpha + \varepsilon_2) \neq p_{\varepsilon_2, \mu}(M\alpha + \varepsilon_2)$ for some $\alpha \in \mathbb{Z}^*$. Hence $h_{1, \mu}(\alpha)$ can not be a polynomial sequence of degree $n_0 + 1$, i.e., there exist $\gamma_0 \in \mathbb{Z}^*$, $|\gamma_0| = n_0 + 2$ and $\alpha \in \mathbb{Z}^*$ such that

$$\Delta^{\gamma_0} h_{1, \mu}(\alpha) \neq 0. \quad (3.8)$$

On the other hand, relation (1.6) tells us that for $\alpha \in \mathbb{Z}^*$

$$h_{k, \mu}(\alpha) = \sum_{\beta \in \mathbb{Z}^*} \sum_{\delta \in \mathbb{Z}^*} a(\alpha - M\beta - M\delta)a_{k-1}(\delta)(\beta + \delta - \delta)^\mu$$

$$= \sum_{\nu \leq \mu} \binom{\mu}{\nu} (-1)^{\mu-\nu} h_{1, \mu}(\alpha) \sum_{\delta \in \mathbb{Z}^*} a_{k-1}(\delta)\delta^{\mu-\nu}$$

$$= \sum_{\nu \leq \mu} \binom{\mu}{\nu} (-1)^{\mu-\nu} h_{1, \mu}(\alpha) \sum_{\delta \in \mathbb{Z}^*} a_{k-1}(\delta)\delta^{\mu-\nu} + h_{1, \mu}(\alpha) \sum_{\delta \in \mathbb{Z}^*} a_{k-1}(\delta).$$
Since \( \sum \limits_{\delta \in \mathbb{Z}} a(\delta) = m \) (see (1.3)), a simple induction argument gives
\[
\sum_{\delta \in \mathbb{Z}} a_k(\delta) = m^{k-1}.
\]
Thus,
\[
\Delta^\gamma h_{k, \mu}(a) = m^{k-1} \Delta^\gamma h_{1, \mu}(a) \quad \forall a \in \mathbb{Z}^n.
\tag{3.9}
\]

It is easily seen by induction that \( \text{supp} \, \Delta^\gamma a_k \subseteq \{ a \in \mathbb{Z}^n : \|a\|_\infty \leq \kappa m^{k/s} \} \) for some constant \( \kappa \) independent of \( k = 1, 2, \ldots \). For any fixed \( a \in \mathbb{Z}^n \), let \( \Gamma_k = \Gamma_k(a) := \mathbb{Z}^n \cap M^{-1}(a - \text{supp} \, \Delta^\gamma a_k) \), i.e., \( \Gamma_k \) denotes the support of \( \Delta^\gamma a_k(a - M^\gamma) \). Then the cardinality of \( \Gamma_k \) satisfies
\[
\# \Gamma_k \leq \kappa' m^k, \quad k = 1, 2, \ldots,
\]
where \( \kappa' \) is a constant. For \( |\mu| = n_0 + 1 \) and \( |\gamma| = n_0 + 2 \) it follows from Hölder’s inequality that
\[
|\Delta^\gamma h_{k, \mu}(a)| = \left| \sum_{\beta \in \Gamma_k} \beta^\mu \Delta^\gamma a_k(a - M\beta) \right|
\leq \left( \sum_{\beta \in \Gamma_k} \|\beta\|_\infty^{(n_0 + 1)/q} \right)^{1/q} \left( \sum_{\beta \in \Gamma_k} |\Delta^\gamma a_k(a - M\beta)|^{p} \right)^{1/p}
\leq c_1 m^{k(n_0 + 1)/s} m^{k/q} \|\Delta^\gamma a_k\|_p
\]
for some constant \( c_1 \) dependent of \( a \) but not of \( k = 1, 2, \ldots \), where \( q \) satisfies \( 1/p + 1/q = 1 \). This together with (2.13) and (3.9) gives us that \( |\Delta^\gamma h_{1, \mu}(a)| \) tends to zero for \( k \to \infty \), in contradiction with (3.8). This completes the induction process, thereby proving the assertion. \( \square \)

**Proof of Theorem 2.3.** The necessity of (2.13) for convergence of the cascade algorithm has already been shown in Theorem 3.2. In order to show sufficiency, we need to prove that conditions (1) and (2) of Result 2.1 follow from (2.13). By Lemma 3.3, the sum rules of order \( n + 1 \) are satisfied. Further, by (2.6) and the definition of \( V_n \) in (2.7) we have for \( |\mu| = n + 1 \)
\[
\lim_{k \to \infty} \|\Delta^\mu a_k\|_p^{1/k} = \lim_{k \to \infty} \|A^k \Delta^\mu \delta\|_p^{1/k} = \rho_p(\{ A_x | V_n \cap \ell(K), \, \epsilon \in E \})
\]
(see [11], Theorem 2.5). Hence, the assertion follows. \( \square \)

Finally we obtain

**Corollary 3.4.** Assume that the sequence \( (Q^k_{\phi_0, n})_{k \geq 1} \) is bounded in \( W^n_p(\mathbb{R}^n) \)-norm. Then the cascade algorithm corresponding to mask \( \phi \) converges for every \( \phi \in W^{n-1}_p(\mathbb{R}^n) \) in \( W^{n-1}_p(\mathbb{R}^n) \)-norm.

**Proof.** Comparing (3.6) with (2.13) (for \( n - 1 \) instead of \( n \)), the assertion directly follows from Theorem 2.3. \( \square \)
§ 4. Perturbations of refinement masks

In this section we shall show the convergence of the cascade algorithm corresponding to a slightly perturbed refinement mask. Moreover, the perturbation of the refinable limit function affected by the perturbation of refinement mask is studied.

**Theorem 4.1.** Let $\Omega$ be a finite set of $\mathbb{Z}^n$. Assume that the cascade algorithm corresponding to $a \in \ell(\Omega)$ converges for every $\phi \in W_n$ in $W_p^n(\mathbb{R}^n)$-norm. Then there is a positive number $\eta$ such that, for any $b \in \ell(\Omega)$ satisfying (1.3), sum rules of order $n+1$ and $\|a - b\|_1 < \eta$, the cascade algorithm corresponding to $b$ also converges for every $\phi \in W_n$ in $W_p^n(\mathbb{R}^n)$-norm.

**Proof.** Recall that $K$ is defined in (2.8). By assumption on $a$, it follows from Result 2.1, that

$$
\lim_{k \to \infty} \|A^k|_{V_n \cap \ell(K)}\|_p^{1/k} = \inf_{k \geq 1} \|A^k|_{V_n \cap \ell(K)}\|_p^{1/k} < m^{-n/s+1/p}.
$$

Hence, there exists an integer $k \geq 1$ and some positive $t$ such that

$$
\max_{v \in V_n \cap \ell(K)} \sum_{\varepsilon_1, \ldots, \varepsilon_k \in E} \|A_{\varepsilon_k} \cdots A_{\varepsilon_1} v\|^p < m^{-n/s+1/p-t}k^p.
$$

Clearly, for this $k$, there is an $\eta > 0$ satisfying that for any $b \in \ell(\Omega)$ with $\|a - b\|_1 < \eta$ we have

$$
\sum_{\varepsilon_1, \ldots, \varepsilon_k \in E} \|A_{\varepsilon_k} \cdots A_{\varepsilon_1} - B_{\varepsilon_k} \cdots B_{\varepsilon_1}\|^p < m^{-n/s+1/p-t}k^p.
$$

Note that $V_n \cap \ell(K)$ is an invariant subspace of any $A_{\varepsilon}$ and $B_{\varepsilon}$, $\varepsilon \in E$. Consequently,

$$
\max_{v \in V_n \cap \ell(K)} \sum_{\varepsilon_1, \ldots, \varepsilon_k \in E} \left\| (A_{\varepsilon_k} \cdots A_{\varepsilon_1}) v - (B_{\varepsilon_k} \cdots B_{\varepsilon_1}) v \right\|^p < m^{-n/s+1/p-t}k^p.
$$

It follows from the triangle inequality that

$$
\max_{v \in V_n \cap \ell(K)} \sum_{\varepsilon_1, \ldots, \varepsilon_k \in E} \|B_{\varepsilon_k} \cdots B_{\varepsilon_1} v\|^p < m^{-n/s+1/p-t_1}k^p,
$$

where the positive number $t_1$ is defined by $2m^{-n/s+1/p-t}k^p = m^{-n/s+1/p-t_1}k^p$. Equality (2.2) tells us now

$$
\rho_p\left( \{ B_{\varepsilon} \mid V_n \cap \ell(K) : \varepsilon \in E \} \right) \leq m^{-n/s+1/p-t_1} < m^{-n/s+1/p},
$$

and the assertion follows from Result 2.1. \hfill \Box

So, in fact, the convergence of $(Q^k_b \phi)_{k \geq 0}$ follows readily from the continuity of the joint spectral radius $\rho_p$. 

Our goal is now to estimate the perturbation of the limit function in terms of the perturbation of the mask, i.e., we want to show that

$$\|Q^k a \phi_0 - Q^k b \phi_0\|_{W^p_p(\mathbb{R}^s)} \leq c \|a - b\|_1, \quad k = 1, 2, \ldots,$$

where $a, b$ meet the assumptions of Theorem 4.1. We use the initial function $\phi_0$ defined in (3.4). Then, choosing $g = Q^k_a \phi_{0,n} - Q^k_b \phi_{0,n}$, the second inequality in (3.5) implies that

$$\sum_{|\mu| = n} \|D^\mu Q^k_a \phi_{0,n} - D^\mu Q^k_b \phi_{0,n}\|_p \leq c m(n/s - 1/p)k \sum_{|\mu| = n} \|\Delta^\mu a_k - \Delta^\mu b_k\|_p.$$

Hence we have to estimate the norm $\|\Delta^\mu a_k - \Delta^\mu b_k\|_p$ for $|\mu| = n$.

In order to obtain this estimate we first need

**Lemma 4.2.** Assume that the masks $a, b \in \ell_0(\mathbb{Z}^s)$ satisfy (1.3) and the sum rules of order $n + 1$ in (2.9). Then for any $v \in V_n$ we have

$$(B_\varepsilon - A_\varepsilon)v \in V_n \quad \forall \varepsilon \in E.$$

**Proof.** We claim that, for any $a$ satisfying sum rules of order $n + 1$ and any $p \in \Pi_n$, there is a polynomial $q \in \Pi_{n-1}$ such that

$$\sum_{a \in \mathbb{Z}^s} p(-\alpha)a(\varepsilon + M\alpha - \beta) = p(M^{-1}(\varepsilon - \beta)) + q(\varepsilon - \beta) \quad \forall \varepsilon \in E \text{ and } \forall \beta \in \mathbb{Z}^s. \quad (4.2)$$

In fact, it follows from Taylor’s formula that

$$p(-\alpha) = \sum_{|\mu| \leq n} \frac{D^\mu p(M^{-1}(\varepsilon - \beta))}{\mu!}(-M^{-1}(M\alpha - \beta + \varepsilon))^{\mu}.$$

Therefore

$$\sum_{a \in \mathbb{Z}^s} p(-\alpha)a(\varepsilon + M\alpha - \beta)$$

$$= \sum_{|\mu| \leq n} \frac{D^\mu p(M^{-1}(\varepsilon - \beta))}{\mu!} \sum_{a \in \mathbb{Z}^s} (-M^{-1}(M\alpha - \beta + \varepsilon))^{\mu} a(\varepsilon + M\alpha - \beta).$$

Note that $a$ satisfies (1.3) and (2.9), i.e., we have $\sum_{a \in \mathbb{Z}^s} a(\varepsilon + M\alpha - \beta) = 1$ and

$$\sum_{a \in \mathbb{Z}^s} (-M^{-1}(M\alpha - \beta + \varepsilon))^{\mu} a(\varepsilon + M\alpha - \beta) = \sum_{a \in \mathbb{Z}^s} (-\alpha)^{\mu} a(M\alpha), \quad |\mu| \leq n,$$
for all $\varepsilon \in E$ and $\beta \in \mathbb{Z}^s$. Hence, we obtain

$$
\sum_{\alpha \in \mathbb{Z}^s} p(-\alpha) a(\varepsilon + M\alpha - \beta) = p(M^{-1}(\varepsilon - \beta)) + \sum_{\alpha < \mu \leq \beta \in \mathbb{Z}^s} \sum_{\alpha \in \mathbb{Z}^s} \frac{D^\mu p(M^{-1}(\varepsilon - \beta))}{\mu!} (-\alpha)^\mu a(M\alpha).
$$

This proves (4.2). Using (4.2) for $b$ instead of $a$ we get a polynomial $g \in \Pi_{n-1}$ such that

$$
\sum_{\alpha \in \mathbb{Z}^s} p(-\alpha)(b(\varepsilon + M\alpha - \beta) - a(\varepsilon + M\alpha - \beta)) = g(\varepsilon - \beta) \quad \forall \varepsilon \in E \quad \text{and} \quad \forall \beta \in \mathbb{Z}^s.
$$

For any $v \in V_{n-1}$ and for any $p \in \Pi_n$, it follows by (2.4) that

$$
\sum_{\alpha \in \mathbb{Z}^s} p(-\alpha)(B_{\varepsilon} - A_{\varepsilon}) v(\alpha) = \sum_{\beta \in \mathbb{Z}^s} g(\varepsilon - \beta) v(\beta) = 0 \quad \forall \varepsilon \in E.
$$

The proof is complete.

**Lemma 4.3.** Suppose that $\Omega \subseteq \mathbb{Z}^s$ is a finite set and that the cascade algorithm corresponding to $a \in \ell(\Omega)$ converges for every $\phi \in W_n$ in $W_\mu^p(\mathbb{R}^n)$-norm. Further, let $b \in \ell(\Omega)$ satisfy (1.3), the sum rules of order $n + 1$ and $\|a - b\|_1 < \eta$, where $\eta$ is chosen such that the assertion of Theorem 4.1 holds. Then there is a positive number $c$ such that we have

$$
\|\Delta^\mu a_k - \Delta^\mu b_k\|_p \leq c\|a - b\|_1 m_n((-n/s+1/p)^k \quad \forall |\mu| = n \quad \text{and} \quad k = 1, 2, \ldots,
$$

where $c$ is independent of $b$ and $k$.

**Proof.** Let $K$ be given in (2.8). By (2.6) and the equality

$$
B_{z_{k}} \cdots B_{z_{1}} - A_{z_{k}} \cdots A_{z_{1}} = \sum_{j=1}^{k} B_{z_{k}} \cdots B_{z_{j+1}}(B_{z_{j}} - A_{z_{j}})A_{z_{j-1}} \cdots A_{z_{1}}
$$

we obtain

$$
\|\langle b_k - a_k \rangle \ast v\|_p
$$

$$
= \left( \sum_{\varepsilon_1, \ldots, \varepsilon_k \in E} \sum_{\gamma \in K} |B_{\varepsilon_1} \cdots B_{\varepsilon_k} v(\gamma) - A_{\varepsilon_1} \cdots A_{\varepsilon_k} v(\gamma)|^p \right)^{1/p}
$$

$$
\leq \sum_{j=1}^{k} \left( \sum_{\varepsilon_1, \ldots, \varepsilon_k \in E} \sum_{\gamma \in K} |B_{\varepsilon_1} \cdots B_{\varepsilon_{j+1}}(B_{\varepsilon_j} - A_{\varepsilon_j})A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_{k}} v(\gamma)|^p \right)^{1/p},
$$

where we have used that by Lemma 2.2 there is some integer $k_0 > 0$ such that both $A_{\varepsilon_1} \cdots A_{\varepsilon_k} v$ and $B_{\varepsilon_1} \cdots B_{\varepsilon_k} v$ are in $\ell(K)$ for all $k \geq k_0$. 

Thus,

\[ \| \left( b_k - a_k \right) \ast v \|_p \leq \sum_{j=1}^{k} \left( \sum_{\varepsilon_1, \ldots, \varepsilon_j \in E} \| B_{\varepsilon_1} \cdots B_{\varepsilon_{j+1}} (B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} v \|_p \right)^{1/p}. \]

Let \( |\mu| = n \). Note that \( \| \Delta^\mu a_{j-1} \|_p = \sum_{\varepsilon_1, \ldots, \varepsilon_{j-1} \in E} \| A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \varepsilon \|_p \). Hence, by (3.6) in Theorem 3.2, there is a constant \( c_1 > 0 \) such that for any \( j \)

\[ \sum_{\varepsilon_1, \ldots, \varepsilon_{j-1} \in E} \| A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \varepsilon \|_p \leq c_1 m\left(\frac{-n}{s+1}+\frac{1}{p}\right)(k-1) \]

(4.4)

On the other hand, from \( \Delta^\mu \varepsilon \in V_{n-1} \) and Lemma 4.2 it follows that

\[ (B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \varepsilon \in V_n \quad \forall \varepsilon_1, \ldots, \varepsilon_j \in E. \]

Moreover, by Theorem 4.1 we already know that the cascade algorithm corresponding to \( b \) converges in \( W^n_p(\mathbb{R}^s) \)-norm and by (4.1) there are a positive number \( t_1 \) and a constant \( c_2 \) such that

\[ \sum_{\varepsilon_1, \ldots, \varepsilon_{j+1} \in E} \| B_{\varepsilon_k} \cdots B_{\varepsilon_{j+1}} (B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \varepsilon \|_p \]

\[ \leq c_2 m\left(\frac{-n}{s+1}+\frac{1}{p}-t_1\right)(k-j) \| a - b \|_1 \quad \forall k > j. \]

This together with (4.4) implies by \( \| B_{\varepsilon_j} - A_{\varepsilon_j} \|_p \leq \| a - b \|_1 \)

\[ \sum_{\varepsilon_1, \ldots, \varepsilon_{j-1} \in E} \sum_{\varepsilon_{j+1}, \ldots, \varepsilon_j \in E} \| B_{\varepsilon_k} \cdots B_{\varepsilon_{j+1}} (B_{\varepsilon_j} - A_{\varepsilon_j}) A_{\varepsilon_{j-1}} \cdots A_{\varepsilon_1} \Delta^\mu \varepsilon \|_p \]

\[ \leq c_3 m\left(\frac{-n}{s+1}+\frac{1}{p}-t_1\right)(k-j) \| a - b \|_1 \quad \forall k > j. \]

where \( c_3 \) is some constant which is independent of \( b \) and \( k \). It follows from (4.3) that

\[ \| (b_k - a_k) \ast \Delta^\mu \varepsilon \|_p \leq c_3^{1/p} \| a - b \|_1 \left(\frac{-n}{s+1}+\frac{1}{p}\right) \sum_{j=1}^{k} \left( \frac{m-(k-j)}{m^{n/(s+1)/p} \sum_{j=1}^{k} m^{-(k-j)1}} \right), \quad k = 1, 2, \ldots. \]

Hence the assertion follows.

We are now ready to present the main theorem of this section.

**Theorem 4.4.** Let \( \Omega \) be a finite set in \( \mathbb{Z}^s \). Assume that the cascade algorithm corresponding to \( a \in \ell(\Omega) \) converges for every \( \phi \in W^n \) in \( W^n_p(\mathbb{R}^s) \)-norm. Then there exists a positive constant \( \eta \) such that, for any \( b \in \ell(\Omega) \) satisfying (1.3) and the sum rules of order \( n+1 \) with \( \| a - b \|_1 < \eta \), the cascade algorithm corresponding to
\( b \) converges for every \( \phi \in W_n \) in \( W^n_p(\mathbb{R}^*) \)-norm. Moreover, there exists a constant \( c \), which is independent of \( b \) and \( k \), such that
\[
\| Q^k_a \phi_{0,n} - Q^k_b \phi_{0,n} \|_{W^n_p(\mathbb{R}^*)} \leq c \| a - b \|_1 \quad k = 1, 2, \ldots
\]  
(4.5)

where \( \phi_{0,n} \) is given in (3.4). Consequently, we find for the limit functions
\[
\| \phi_a - \phi_b \|_{W^n_p(\mathbb{R}^*)} \leq c \| a - b \|_1.
\]  
(4.6)

Proof. By Theorem 4.1 we know that for \( b \in \ell(\Omega) \) satisfying the sum rules of order \( n + 1 \) and with \( \| a - b \| < \eta \) for some suitable \( \eta \) the cascade algorithm corresponding to mask \( b \) converges for every \( \phi \in W_n \) in \( W^n_p(\mathbb{R}^*) \)-norm. Therefore, \( \phi_b \in W^n_p(\mathbb{R}^*) \).

Since \( \phi_{0,n} \in W_n \), the cascade algorithm converges for \( \phi_{0,n} \) for \( a \) and \( b \), i.e., we have
\[
\lim_{k \to \infty} \| Q^k_a \phi_{0,n} - \phi_a \|_{W^n_p(\mathbb{R}^*)} = \lim_{k \to \infty} \| Q^k_b \phi_{0,n} - \phi_b \|_{W^n_p(\mathbb{R}^*)} = 0.
\]

The inequality (4.6) follows now from (4.5).

In order to prove (4.5) we appeal to Theorem 3.2. Put \( \lambda = \Delta^\mu a_k - \Delta^\mu b_k \) in (3.5). This corresponds to \( g = Q^k_a \phi_{0,n} - Q^k_b \phi_{0,n} \). Then the second inequality in (3.5) yields for some constant \( c_1 \) and for \( k = 1, 2, \ldots \)
\[
\sum_{|\nu|=n} \| D^\nu (Q^k_a \phi_{0,n} - Q^k_b \phi_{0,n}) \|_p \leq c_1 m(n/s-1/p)k \sum_{|\nu|=n} \| \Delta^\mu a_k - \Delta^\mu b_k \|_p.
\]

Together with Lemma 4.3, it in turn implies
\[
\sum_{|\nu|=n} \| D^\nu (Q^k_a \phi_{0,n} - Q^k_b \phi_{0,n}) \|_p \leq c_2 \| a - b \|_1, \quad k = 1, 2, \ldots
\]  
(4.7)

where \( c_2 \) is some positive number being independent of \( b \) and \( k \).

As shown in Corollary 3.4, the cascade algorithm corresponding to \( a \) also converges for every \( \phi \in W_{n'} \) in \( W^n_p(\mathbb{R}^*) \)-norm with \( n' < n \). Replacing \( n \) with \( n' \) in (4.7) and then taking the sum of the resulting inequalities we obtain (4.5).

We obtain the following corollary.

**Corollary 4.5.** Let \( \Omega \) be a finite set in \( \mathbb{Z}^s \). Suppose that \( \phi_a \) is a refinable function in \( W^n_p(\mathbb{R}^*) \) corresponding to mask \( a \in \ell(\Omega) \) and the shifts of \( \phi_a \) are stable. Then there are positive constants \( \eta \) and \( c \) such that, for any \( b \in \ell(\Omega) \) satisfying (1.3), the sum rules of order \( n + 1 \) and \( \| a - b \|_1 < \eta \), the refinable distribution \( \phi_b \) is in \( W^n_p(\mathbb{R}^*) \) and satisfies (4.6).

Proof. By the stability of the shifts of \( \phi_a \), the cascade algorithm corresponding to \( a \) converges on \( W_n \) in \( W^n_p(\mathbb{R}^*) \)-norm. This conclusion has been established in [19] for \( p = 2 \). The method works for general \( p \geq 1 \). Now, using Theorem 3.2, the proof is analogous to that of Theorem 4.4. \( \Box \)
Remark. The proof of the estimate (4.5) is strongly based on the second inequality in (3.5). This inequality in turn has been shown for our initial function \( \phi_0 \) in (3.4) using the relation (3.2). Since not every function \( f \) in \( W_n \) satisfies the relation \( D^\mu f = \Delta^\mu g \) for some suitable \( g \) as in (3.2), the arguments in this paper fail to work for a general initial function in \( W_n \). This difficulty has been overcome recently by Han [9]. In this paper, he established inequality (4.5) for any initial function in \( W_n \).

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