Cramér-Rao-Bound for Coherent Dual-Band Radar Range Estimation

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Abstract—This paper investigates accuracy bounds on frequency estimation from dual-band observations as they may be obtained from multi-sensor platforms with several radar sensors operating in different frequency ranges. The Cramér-Rao lower bound is derived and the resulting minimum achievable resolution is given as a function of the frequency lag between the observation bands. It is seen that the theoretical resolution bound decreases as the frequency lag increases.

An iterative nonlinear least-squares estimation algorithm is used to coherently process dual-band radar information. Monte-Carlo simulations show that this estimator achieves smaller variance with dual-band observations as compared to equal bandwidth single-band observations.

I. INTRODUCTION
With increasing tendency, platforms with a variety of different radar sensors operating in different frequency regimes become available. Very often, a part of these sensors offer information about the same target scenery. It is a fundamental fact in estimation theory that any additional sensor improves the maximum reachable estimation accuracy. Provided sufficient sensor properties and processing algorithms, the signal energy as well as the bandwidth delivered from the individual sensors may be combined, making the totally covered bandwidth and the sum of the signal energies available. Classical radar signal theory predicts the gain resulting from proper use of bandwidth and signal energy.

The question remains whether there is also benefit if sensors provide frequency-domain information about the same scenery but at significantly distinct carrier frequencies. Can we expect better accuracy and resolution from the fact that there is a certain frequency difference between distinct measurement bands?

It is well known that radar range information can be represented in time domain and equivalently in frequency domain. Conversion between the impulse response (the actual range profile) and the frequency response of the complex reflectivity is generally possible by means of the Fourier transform. A point target with round trip time $\tau$ corresponds to a complex sinusoid in the frequency response. Detecting and locating radar point targets in frequency domain is thus equivalent to detecting exponentials in noisy measurements and estimating their parameters, the frequencies in particular. Therefore, the problem is at first reduced to estimating the frequencies of sinusoids in noise from multiple observation vectors. The resulting effect of a considerable distance between the observation windows along the sample axis on theoretical estimation accuracy bounds is of particular interest. It is shown that an important prerequisite is mutual coherency between the used measurement data. If mutual phase information can be employed, then improved frequency estimation accuracy is possible through a lag between the data windows.

II. DUAL-BAND RADAR DATA
In the following we assume that the time-domain radar response consists of $P$ discrete round trip time $\tau_p$ and magnitude $A_p$. The idealized impulse response is then

$$h_0(t) = \sum_{p=1}^{P} A_p \delta(t - \tau_p)$$

where $\delta(t)$ is the Dirac delta pulse. With regard to the equivalence of information in time-domain and in frequency-domain we continue our considerations in frequency-domain. The corresponding frequency-domain complex reflectivity of an ideal multipath radar scenery is given by

$$r(f) = \sum_{p=1}^{P} A_p e^{-j2\pi f \tau_p}.$$ (2)

Let us further assume that this complex reflectivity is sampled at $L_n$ discrete and equidistant frequencies within a frequency band $B_n$ with start frequency $\omega_{0n}$ and bandwidth $B_n$. With $\omega = \omega_{0n} + k\Delta\omega$ the resulting series of discrete reflectivity values is then

$$r_n[k] = \sum_{p=1}^{P} a_{pn} z_p^k ; \quad k = 0, 1, 2, \ldots, L_n - 1.$$ (3)

In this expression it is $z_p = e^{-j\Omega_p}$ with $\Omega_p = \Delta\omega \tau_p$ and

$$a_{pn} = A_p e^{-j\phi_{pn}}$$ (4a)

with

$$\phi_{pn} = \omega_{0n} \tau_p = \frac{\omega_{0n} \Delta\omega}{\Delta \phi_{pn}} \Omega_p = k_0n \Omega_p.$$ (4b)

For the series $r_n[k]$ and $r_m[k]$ in two subbands $B_n$ and $B_m$ the relation

$$\frac{a_{pn}}{a_{pm}} = e^{-j\beta_p(k_{0n} - k_{0m})}$$ (5)

follows. Equation (5) states the phase relation between subbands $B_n$ and $B_m$, and it is thus termed the coherency relation. Provided the same point targets are active in frequency regimes $B_n$ and $B_m$, (5) can be used during the process of estimating the values $\Omega_p$ from multiband radar reflectivity measurements.

Let us now restrict to the case of two subbands. The parameters of the assumed input data are illustrated in fig. 1. Two radar sensors with signal bandwidths $B_{1,2}$ provide information about the complex reflectivity of the radar scene. The variable $k$ denotes the index of discrete frequency measurement points and $k_{01,02}$ correspond to the start frequencies of both measurement bands, respectively. In general, the available frequency bands do not adjoin each other and therefore, the distance between the bands is specified by the number $D$ of unavailable samples in between.
In the following, the process of estimating the frequencies of sinusoids in dual-band reflectivity measurements is under investigation. It is assumed that the sinusoid frequencies and magnitudes are identical in both subbands and thus, the same scattering centers are observed by the two radar sensors. Since the sinusoid frequencies directly correspond to the target ranges, the maximum achievable frequency estimation accuracy for the dual-band case is of particular interest in this contribution.

III. CRAMÉR-RAO LOWER BOUND FOR DUAL-BAND FREQUENCY ESTIMATION

Let us introduce a noisy measurement

\[ x[k] = s[k; \theta] + w[k], \quad k = 0, 1, \ldots, L - 1 \]  

where \( s \) is the noiseless signal that depends on the vector parameter \( \theta \) and \( w \) is white Gaussian noise. Following [1], the Fisher information matrix is given by

\[ [I(\theta)]_{ij} = \frac{1}{\sigma^2} \sum_{k=0}^{L-1} \frac{\partial s[k; \theta]}{\partial \theta_i} \frac{\partial s[k; \theta]}{\partial \theta_j} \]  

and the Cramér-Rao lower bound (CRLB) for estimation of the \( i \)-th parameter is

\[ \text{var}\{\theta_i\} \geq [I^{-1}(\theta)]_{ii}. \]  

Now the maximum achievable range accuracy and range resolution using dual-band measurements are of question. It is of particular interest whether there is a possible processing gain due to the fact that two measurement frequency ranges are not adjacent but \( D \) samples apart from each other. This gain would then lie beyond the classically predictable gain that results from the sum of the bandwidths and, in the coherent case, from the sum of the signal energies.

A. Single frequency estimation

First we consider the signal

\[ s[k; \theta] = \cos(2\pi f_0 k + \phi) \]  

observed in two windows where

\[ k = k_{01}, \ldots, k_{01} + L_1 - 1 \] in \( B_1 \)
\[ k = k_{02}, \ldots, k_{02} + L_2 - 1 \] in \( B_2 \)

respectively. The parameter vector is \( \theta = [f_0 \phi]^T \) and \( f_0 \) is to be estimated. The CRLB for this case is shown in fig. 2 for increasing values of \( D \). The length of the data windows are \( L_1 = L_2 = 5 \) and the signal-to-noise ratio is \( \text{SNR} = 0 \text{dB} \). This plot does not take into account the phase relation between \( B_1 \) and \( B_2 \), because the phase \( \phi \) in \( B_2 \) is assumed to be independent of \( f_0 \) and \( \phi \) in \( B_1 \). Hence, this is termed the non-coherent case. The CRLB tends to infinity as \( f_1 \) approaches DC or the Nyquist limit. This is because at those frequencies a slight change in \( f_1 \) does not alter the observation (10a) very much. It can be seen that the local maxima of the CRLB vary, dependent on which part of a period is observed in the two windows. However there’s no significant decrease of the CRLB with increasing gap \( D \).

We now write the two observations in the form

\[ s_1[k] = \cos(2\pi f_0 k + \phi_1), \quad k = 0, 1, \ldots, L_1 - 1 \]  
\[ s_2[k] = \cos(2\pi f_0 k + \phi_2), \quad k = 0, 1, \ldots, L_2 - 1 \]  

and take into account that \( \phi_2 = \phi_1 - 2\pi f_0 (L_1 + D) \). The parameter vector is now \( \theta = [f_0 \phi_1]^T \) and again, \( f_0 \) is to be estimated given the two observations (10). The resulting CRLB is shown in fig. 3. As this result now uses the phase relation between the data in two distinct windows, this is termed the coherent case. Because \( |\phi_2| / f_0 \) increases with \( D \), the CRLB now becomes smaller as the distance \( D \) becomes larger. Note, that the total number of used samples \( L_1 + L_2 \) remains constant in figs. 2 and 3.

B. Dual frequency estimation

In order to investigate the maximum achievable resolution, we start with the observations

\[ s_1[k; \theta] = \cos(2\pi f_1 k + \phi_1) + \cos(2\pi f_2 k + \phi_2) \]  
\[ s_2[k; \theta] = \cos(2\pi f_1 k + \phi_{12}) + \cos(2\pi f_2 k + \phi_{22}) \]

of a signal containing two sinusoids with unknown normalized frequencies \( f_1 \) and \( f_2 \). The phases \( \phi_{12} \) are also considered unknown and so the parameter vector is given by \( \theta = [f_1 f_2 \phi_1 \phi_2]^T \). Since (11a,b) are observations of the same

\[ \text{var}\{\theta_i\} \geq [I^{-1}(\theta)]_{ii}. \]
signal, it follows that

\[ \phi_{12} = \phi_1 - 2\pi f_1(L_1 + D) \]  
\[ \phi_{22} = \phi_2 - 2\pi f_2(L_1 + D) \]

with \(L_1\) and \(D\) as illustrated in fig. 1. Extending (7) to two bands yields

\[
[I(\theta)]_{ij} = \frac{1}{\sigma^2} \left( \sum_{B_1} \frac{\partial s_1[k; \theta]}{\partial \theta_i} \cdot \frac{\partial s_1[k; \theta]}{\partial \theta_j} + \sum_{B_2} \frac{\partial s_2[k; \theta]}{\partial \theta_i} \cdot \frac{\partial s_2[k; \theta]}{\partial \theta_j} \right)
\]

where the partial derivatives in \(B_2\) take into account the coherency property (12). From (13) the CRLB for estimating frequencies \(f_1\) and \(f_2\) are

\[
\text{var}\{f_1\} \geq [I^{-1}(\theta)]_{11}
\]
\[
\text{var}\{f_2\} \geq [I^{-1}(\theta)]_{22}
\]

respectively. A plot of (14a) for \(f_2 = 0.25\) is shown in fig. 4. In this figure, both subbands have equal length \(L_1 = L_2 = 5\) and signal-to-noise is \(\text{SNR} = 0\ \text{dB}\). As a parameter, the interband gap \(D\) varies. In the case \(D = 0\) a single band with length \(L = 10\) is used. In addition to the singularities at DC and its image, the minimum achievable variance for \(f_1\) increases when \(f_1\) comes close to \(f_2\). It can be seen, however, that the CRLB globally decreases as the interband gap \(D\) increases. At the same time the corridor around \(f_2\) becomes narrower, indicating a better resolution. Since \(I^{-1}(\theta)_{11}\) depends on both \(f_1\) and \(f_2\), this plot includes prior knowledge on the value of \(f_2\). It shows the CRLB for the estimation of \(f_1\) beside a second signal at frequency \(f_2 = 0.25\).

Let us now define the minimum achievable resolution as the frequency distance \(\delta f = |f_2 - f_1|\) where \(\text{var}\{f_1\}\) equals \(\delta f\). In this way we assume, that a second frequency \(f_1\) can be well discriminated from \(f_2\), as long as the estimator’s variance is smaller than the actual difference between \(f_1\) and \(f_2\). This value of \(\delta f\) can be derived from fig. 4 and the result is shown in fig. 5 versus the interband gap \(D\). It can be seen that the achievable resolution improves with increasing \(D\) even when the total available bandwidth \(B_1 + B_2\) remains constant, as it is the case in fig. 5. In this plot \(L_1 + L_2 = 20\) so the classical Rayleigh resolution bound is \(0.5/20 = 250 \cdot 10^{-4}\).

IV. LEAST-SQUARES PARAMETRIC MODEL FITTING

Spectral estimation techniques based on autocorrelation sequence (ACS) estimation can generally process multi-band data because ACS estimation results are not altered by an arbitrary number of zeros padded to the data. In this way, autoregressive parametric models (AR, ARMA) as well as eigenanalysis based methods (MUSIC) are sufficient multi-band frequency estimators. They have to be considered incoherent techniques, however, since they do not depend on the actual frequency lag between the contributing sensors. As a consequence, ACS based methods do not employ phase information that would be available, if the contributing sensors were coherent and if they were illuminating the same radar scenery.

Fitting the exponential model described in (2)–(5) with a common parameter set to both measurement subband data takes into account phase and frequency relationships that are to be expected between the subband data, if they are different measurements of the same scenery. The algorithm used for dual-band frequency estimation uses the root-MUSIC algorithm to estimate initial pole angles. They serve as start values in a global exponential model which derives the complex sinusoid amplitudes through a least-squares fit to the dual-band measurement data. A nonlinear iterative minimization procedure then tunes the pole angles as well as their complex amplitudes to minimize the total squared deviation between the global signal model and the measured subband data. During each minimization cycle, the sinusoid amplitudes are adapted in due consideration of the modified pole angles. As a result we get pole angle estimates that fit better to the observed data than the initial values do. See [2] for a detailed description of the estimation procedure.

Figure 6 shows the variance and the mean of the resulting pole angle estimates versus signal-to-noise (SNR) for the single-band (\(D = 0\)) and for the dual-band (\(D = 128\)) case. The true pole angles are indicated by horizontal dashed lines. Their angular distance corresponds to the classical resolution limit \(\Delta f = 300^\circ/(L_1 + L_2)\). It can be recognized that the variance of the pole angle estimates in the range \(4\ \text{dB} \leq \text{SNR} \leq 20\ \text{dB}\) is significantly smaller for the dual-band case (fig. 6b). For low SNR and for \(\text{SNR} > 30\ \text{dB}\) the quality of the estimates is equivalent to the single-band case.
V. CONCLUSION

The Cramér-Rao lower bound (CRLB) for frequency estimation indicates the possibility to improve accuracy using subwindows with a considerable lag in between them. A measure for the resolution has been derived from the CRLB and it shows that lower estimation variance potentially yields higher resolution. Hence there is a gain in resolution without the need for additional bandwidth when using multi-frequency observations of the same radar scenery. The improved estimation quality is based on taking into account phase information between the used radar subbands. This can generally be done, when the same scatterers are active in all measured frequency ranges. Simulations using a global all-pole model fit to noisy dual-band data show that this procedure can reduce the frequency estimation variance by a factor of 1.5–1.8 in the range of $\Delta \text{SNR} \leq 20 \text{dB}$.

REFERENCES