Heinz Steinbring

The Context for the Concept of Chance – Everyday Experiences in Classroom Interactions

THE CONTEXT FOR THE CONCEPT OF CHANCE – EVERYDAY EXPERIENCES IN CLASSROOM INTERACTIONS

HEINZ STEINBRING

The paper analyses in an exemplary way how informal and formal aspects of the concept of chance are negotiated in classroom interaction. An important theoretical perspective is the relation between theory and practice in probability for getting a deeper insight in the development of the concept of stochastic randomness starting from the informal concept of chance. Two short teaching episodes dealing with games of chance in mathematics teaching are analyzed, illustrating what kind of context for the chance concept is constituted in everyday teaching. During official classroom communication, chance is mainly used as a reduced and methodically generalized notion, despite some students’ critical remarks allowing for a theoretical development of this concept.

1. INTRODUCTION: CHANCE AS AN INFORMAL AND AS A FORMAL CONCEPT

The concept of chance refers mainly to two different domains from where it gains its sense and meaning. On the one hand, there are deeply rooted understandings of chance stemming from social and everyday experience; chance is related to luck, bad luck, fortune etc. On the other hand, during the historical development of probability theory, chance has undergone a transformation and a specialization relating chance to the concept of stochastic independence.

These two interpretations could be characterised the former as an informal and the other as a formal concept of chance:

– Chance as an informal concept: Chance or accident (from the Latin word falling) is a pattern for describing some effect which has happened of which we either do not know the cause or whether there is a true cause for this effect or not. In this way, chance has been used to cope with the cause–effect relations which seem to be too complex, which could not be analyzed or with those situations in which there seemed to be no deterministic relation between cause and effect. In this way, chance first of all evolved as a concept describing non-regular, lawless
phenomena. Chance became the opposite of regular and deterministic events. The concept of chance developed in contrast to the conception of determinism and displayed, for instance, the following various aspects:

- chance as a substitute for deterministic explanations in situations not understood according to traditional causal patterns, (i.e. if it were possible to know and to measure exactly all physical parameters of a die, one should be able to predict the outcomes of throwing this die, an argument also expressed by J. Bernoulli),

- chance as a combination of many different and unequally distributed causes leading to some unforeseen effect, (i.e. describing the chaotic movement of atomic particles with the help of chance and probability),

- chance as an aspect of personal freedom in contrast to the necessity and strictness of deterministic behavior, (i.e. chance introduces yet undecided events and the possibility of personal choices into the course of one's life which according to the Laplacean determinism would be strictly fixed from the very birth)

- chance as a description for the happening of an event with a very low probability, an event which under normal conditions will never happen (i.e. winning in the lottery game the first prize with a very low probability).

According to this interpretation the concept of chance reflects many philosophical, epistemological, social and personal aspects; it relates to world views, to models of interpreting cause and effect patterns and to personal constructs such as luck, bad luck and unforeseen events. It represents a combination of objective and subjective aspects; chance expresses the individual, subjective disability to reveal the assumed deterministic pattern. Here we have an important source for the confrontation between objective and subjective probabilities. (For further interpretations of the informal concept of chance see, for instance, Hacking 1975 & 1990, Hörz 1980, Sachsse 1979).

- **Chance as a formal concept**: During the historical development of probability theory, the concept of chance has changed and has been modified to a formal mathematical concept. (cf. Steinbring 1980, v. Harten & Steinbring 1983). On the one hand this has been a specialization only reflecting some aspects of the informal concept of chance; on the other hand, the specialization has been the starting point for a mathematical generalization allowing for new insights in everyday statistical situations. The central idea of formalizing the concept of chance has been to develop the concept of a chance sequence, or better, of a
random sequence. Conducting a statistical experiment, for instance tossing a die or a coin, results in a sequence of outcomes. This sequence of heads and tails or of the pips of the die is called a random sequence if the single trials are performed independently.

In this way, the formalization of the chance concept makes first of all a distinction between chance and probability by introducing the concept of stochastic independence. (cf. Fine 1973). Independence is defined by the multiplication rule of probabilities, a kind of implicit or theoretical definition; chance is no longer related to small probabilities of events. The concepts of probability and of chance in the shape of stochastic independence are mutually dependent (Kac 1959 & 1982, Steinbring 1991a, 1991b), they are defined reciprocally as theoretical concepts.

A random sequence displaying the formal concept of chance is a sequence of outcomes (for instance numbers) which have been produced independently; the probabilities of the events may differ (being very small or big) and a further variation might be some degree of dependency in the sequel of the trials (Markovian chain). In this way the informal concept of chance transformed to the formal concept of stochastic randomness first of all reflects the concept of independence of the outcomes and makes a differentiation between chance and (small) probabilities. This transformation of the informal to the formal concept at the same time produces a general and theoretical concept of randomness: there is never a definite definition for a given sequence, whether it is really random or not; this can only be tested via statistical tests or randomness and never determined absolutely. This means that the concept of formal randomness is implicit and open. For concrete applications it has to be explored and specifically interpreted. There is no automatic decision about chance or randomness in a given concrete case. (cf. Chaitin 1975, Steinbring 1991a).

On the basis of this distinction between the formal and the informal side of chance and randomness, in the following two short teaching episodes will be analysed to better understand how in everyday classroom interaction the concept of chance is introduced, what types of argumentation are used, in which ways students and teacher establish and negotiate the meaning of chance and randomness and what are the methodological intentions and means of the teacher to justify and to explain this concept.
2. ANALYZING TWO CONTEXTS FOR THE CONCEPT OF CHANCE IN CLASSROOM INTERACTION

In the following, the distinction between the informal concept of chance and the formal concept of randomness shall be explored as it is used in everyday classroom interaction. We shall study two game situations which have been conducted and discussed in a fifth grade (students of age 11); in each case, the experimental results are compared to a formal mathematical explanation, trying to provide a reasonable justification of the difference between the observed empirical outcomes and the theoretically expected probability values. In both cases, the concept of chance is introduced and used to provide this required justification. The first episode is from the second lesson, and the second episode from the third lesson of a first introductory course in elementary probability. The first lesson started with conducting a chance game only slightly different from the one described in the first episode; this was the only experimental background on which the students based their beginning stochastic reasoning. (The following short transcribed statements are all taken from Voigt 1983, pp. 240 – 244, for the first episode, and pp. 271 – 277, for the second episode.)

2.1 DRAWING LITTLE DOLLS FROM AN URN

In the preceding lesson, the students played this game:

In an urn there are 1 yellow, 2 green and 5 red dolls. Rules of the game: One doll will be drawn from the urn and then replaced. For every draw a stake of 10 Pfennig has to be paid. Following gains are possible: When drawing a green doll, 20 Pfennig will be paid and 30 Pfennig when drawing the yellow doll.

The homework for the students was to play this game 20 times and to note the results. At the beginning of the lesson, some results obtained by the students were written down on the blackboard:

blackboard image:

| Urn: 5 red, 2 green and 1 yellow doll |
| Game: Drawing a doll with replacement of the doll |
| Stake: 10 Pf per draw / |
| Gain: 20 Pf when drawing a green doll |
| 30 Pf when drawing the yellow doll |

| Kitty | 20 20 20 20 20 20 20 20 20 20 20 20 20 20 20 20 20 20 20 20 | 200 |
| Amount won | 21 24 26 27 19 20 13 23 25 17 215 |
In this table the following results are noticed: In row 'Kitty', the number of the games played is indicated which results in a stake of 20 times 10 Pfennig; in row 'Amount won', the number of 10-Pfennig pieces is noticed, which have been won playing 20 times the game. All together 200 games are studied, which requires a stake of 200 times 10 Pfennig or 20 Mark; the total gain is 215 times 10 Pfennig or 21,50 Mark. Compared with the stake paid the effective gain is 1,50 Mark in 200 games.

Underneath this table, the student Pascal has noticed his ideas why usually one has to expect a loss playing this game:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2 green</td>
<td>40 Pf</td>
<td>dolls</td>
<td></td>
</tr>
<tr>
<td>1 yellow</td>
<td>30 Pf</td>
<td>8 Games</td>
<td>80 Pf</td>
</tr>
<tr>
<td></td>
<td></td>
<td>kitty</td>
<td>70 Pf</td>
</tr>
</tbody>
</table>

For taking out all eight dolls from the urn one has to pay 8 times 10 Pf, i.e. 80 Pf. But the gain will only result in 70 Pf, 40 Pf for the two green dolls and 30 Pf for the yellow doll.

When discussing this game the student Pascal delivers a piece of an explanatory 'theory'. He takes up an argument the teacher has developed in the lesson before for a similar game situation and he proposes an "ideal mass experiment": Instead of repeating the real experiment many times, Pascal proposes to look at what will happen on the average, without explicitly explaining it in this way. He takes out at once all the eight dolls paying 80 Pfg stakes and winning only 70 Pfg. Upon writing his theoretical explanation on the blackboard, Pascal says: "... and if one compares it now, so 80 Pfennig with 70 Pfennig, then one has lost 10 Pfennig."

Having in mind the experiment with this game in the preceding lesson, the data they have obtained during their homework and the 'theory' delivered by Pascal, teacher and students now start discussing the outcomes of the game.

The main controversial point is the difference between the empirical outcomes and the theoretically expected loss of ca. 10 Pfg. in eight draws. Indeed, one could statistically calculate that the empirical result noted on the blackboard has a very little probability of less than 0.1% according to the Bernoullian model. This inconsistency between the empirical outcomes and the theoretically expected values is talked about between the students and the teacher. The teacher reinforces those contributions of students stating that one should believe in the empirical results, that
even this might happen, that the occurrence of this gain could depend on chance. And in particular this is what the teacher intends – he emphasizes that the chance concept is appropriate to explain this (big) difference between theory and empirical phenomenon. For strengthening his intention, the teacher argues by giving the example of drawing eight times in succession the yellow doll. This can be seen in the following short transcript:

T: what would you say if I drew ... the yellow doll seven times in succession.
S: chance.
S: luck.
S: this is chance.
T: how do you call such a thing'
S: cheating.
T: Hermine.
He: chance.
T: yes, this is chance. By chance, this result comes out. there is chance in that.

The chance concept is introduced and used here to give an explanation for the big difference between empirical outcomes and the theoretical prediction; chance becomes relevant here when a seemingly improbable event nevertheless occurs. The difference here is not taken as a signal to control the game, for instance whether the experiment has been statistically correctly conducted or not. A closer analysis shows that the students did not perform stochastically independent draws in their homework, thus violating the formal concept of randomness; in the preceding lesson, the students performed the similar drawing experiment with replacement, but some of them refused to accept unfavourable outcomes by arguing that the person who made the draw has looked into the urn before the draw. This could be an explaining basis for discussing the stochastic interpretation of the observed values. According to his methodical intentions, the teacher now focusses on a different interpretation.

"In this lesson – and this is the starting point – a contradiction between theoretical prediction and empirical observation is stated: Pascal's elementary theory predicts a loss in the game, whereas the actual playing seemingly results in gain.

How is this contradiction handled? In the end, a methodical reduction is presented as a solution: The reverse-definition of chance as a non-existing regularity has become more and more universal. A theoretically impossible gain has occurred, but because this empirical phenomenon is not subject to any causal law, the standard justification is valid in this case as well: It is 'chance' in the form of 'having good luck' which serves as the conventional explanation for this observed contradiction."

(Steinbring 1991a, 515).
2.2 TOSSING A DIE 1200 TIMES

During the next lesson the students performed another probability experiment; in groups of five students they tossed a die and counted the outcomes. The results of all trials were noted in a table on the blackboard.

<table>
<thead>
<tr>
<th>Pips of the die</th>
<th>Frequency</th>
<th>theoretical Frequency</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>175</td>
<td>200</td>
<td>1 of 6</td>
</tr>
<tr>
<td>2</td>
<td>183</td>
<td>200</td>
<td>1 of 6</td>
</tr>
<tr>
<td>3</td>
<td>229</td>
<td>200</td>
<td>1/6 or 1 of 6</td>
</tr>
<tr>
<td>4</td>
<td>218</td>
<td>200</td>
<td>1 of 6</td>
</tr>
<tr>
<td>5</td>
<td>208</td>
<td>200</td>
<td>1 of 6</td>
</tr>
<tr>
<td>6</td>
<td>187</td>
<td>200</td>
<td>1 of 6</td>
</tr>
</tbody>
</table>

Here again, students and teacher discuss the empirical outcomes in comparison with theoretical arguments and expectations. And it is the teacher who in an non-technical way first asks what could be "expected" and later on this basis directly speaks of the "theoretical expectation of the frequency value". Probability theory provides the ideal values, whereas in practical situations it always looks different. The teacher's intention is to develop the idea of the so-called empirical law of large numbers (without stating this technical term): Increasing the number of trials will result in a "convergence" of the empirical outcomes towards the theoretically expected values, or more exactly: a "convergence" of the relative frequencies towards the ideal probability value.

Ca: No, . . . it must needn't always fit exactly, there is always a deviation of one or two, but, the more games, the closer one comes to the theoretical result.

T.: Hmm, that is, the longer one plays, the oftener one tosses, the smaller is the chance.

NS: (...) T.: The theoretical frequency values fit the experimentally found values, says Carsten.

In the following there is a clarification of the way of how to approach the theoretical results. When increasing the number of trials, the absolute values will not converge, but only the relative values. In this
example the teacher asks the students to calculate the following quotients: Divide the observed frequency values by the theoretical frequency of 200 for each outcome. The two most strongly deviating values of 175 and 229 are taken and divided. The calculated quotient values are estimated to be very close to one.

T.: now, just for fun, divide ...two-hundred-twenty-nine, also a big deviation in that, by two-hundred
NS: (...)  
T.: is the result approximately one, too'  
S.: one point hundred-and-forty-five.  
T.: yes, this number is also very close to one, and these numbers...  
these quotient values then are even closer to one if one tosses two-thousand-four-hundred times, three-thousand-six-hundred times, one gets closer and closer to the one.

Under these circumstances, it is a real statistical problem to estimate whether the observed outcomes deviate significantly from the theoretical expectations. For the total distribution the $\chi^2$ coefficient is 11.56 which implies a rejection (on the 95% level with five degrees of freedom) that the die is an ideal one. And if one restricts to the two outcomes "pip of 3" and "pip of non-3" (with probabilities 1/6 and 5/6) the Binomial model shows that there is only a small probability of ca. 1.5% for getting in 1200 trials 229 or more times the pip of 3 when tossing an ideal die. Statistically, the calculated quotient 1.145 is not close to one. Also, in this case there is a significant deviation between the observed and the expected values. But now the statistical example should serve to show how chance can be eliminated and how the deviations produced by chance disappear.

T.: ... well, the theoretical probability value, the probability value is purely theoretical, which is attained if one can eliminate chance. And chance can be eliminated if one plays very long. If one only plays briefly, chance can be very big. But if one tosses the dice ten-thousand times, one will certainly have a more regular distribution. Chance has thus been eliminated.

Against this interpretation of the teacher one student argues that chance will never be eliminated; this would exactly be chance if there were a total coincidence between empirical and theoretical values:

St: ... I think it again depends on chance, whether everything is the same, whether everything has been eliminated, then eh... eh, that can be never calculated, cant it. Perhaps the six will have been tossed once more than the five or so. But (...)  
T.: you are nevertheless of the opinion...  
St: totally eliminated, it is chance if chance is eliminated.
the number of trials, the smaller the chance, i.e. the deviations between empirical outcomes and theoretical values. Chance could be totally eliminated, if one could perform infinitely many trials in an ideal way.

3. MEANING AND CONTEXT FOR THE CHANCE CONCEPT IN CLASSROOM INTERACTION

What kind of notion of chance is dominant and developed interactively between teacher and students? There seems to be a link to the informal concept of chance in the way to explain chance by the degree of deviation between empirical results and theoretical expectations. If there are large deviations to be observed, then chance rules; and if the number of trials increases, chance will vanish.

The explanations and justifications as negotiated during the first episode could be summarized in the following type of argument:

If there is a (large) difference between theory and empirical facts, chance is strongly governing. This argument is supported by the teacher’s example of drawing the yellow doll seven times in succession, a very improbable event which could only happen by chance or by great luck.

The explanations and justifications as negotiated during the second episode could be summarized in the following type of argument:

If the number of trials were increased, and if one could perform ideally infinitely many trials, then chance could be decreased or even totally be eliminated. Then there would be no more difference between theoretical and empirical values.

Chance as conceived of in this classroom interaction becomes an explanatory description for the improbable, for events which have a low probability and occur only rarely and which therefore display a large deviation of empirical outcomes from theoretical expectations. The first argument states the conditions under which chance exists, the second explains conditions for eliminating chance, that is for the absence of chance.

In this way, the concept of chance constituted in the classroom is linked to the informal ideas of luck, fortune and misfortune. What is the implicit foundation on which this idea of chance is based? An important aspect is the distinction between "theory" and "practice": the practical side is represented by exemplary games, tossing a die and drawing dolls. How are these experiments assessed? There is much evi-
dence in the course of classroom interaction that these games are not taken seriously in a strict sense. They are taken by the teacher as a kind of "idealized chance experiments": they are abstracted from the concrete experimental level. The temporary and situated particularities are not taken into account, the game is already thought of as an ideal one, fulfilling the presupposed requirements of equal probabilities of the dice or the urn and the presupposed condition of stochastic independence.

According to this conception of the idealized game, one has to distinguish between an ideal and a concrete chance experiment, which are differently related to the "theoretical" side.

![Diagram 1]

All the arguments, intentions and explanations discussed and elaborated during the two episodes seem first of all to be negotiated on the level between the ideal experiment and probability theory. The ideal experiment already fulfills all requirements. There is no real question about the empirical experiment, about the concrete chance situation.

But if one really wants to explore the concrete situation and better understand its particular features and conditions, one has to adopt another perspective concerning the relation between theory and practice in elementary probability theory.

![Diagram 2]
But if one really wants to explore the concrete situation and better understand its particular features and conditions, one has to adopt another perspective concerning the relation between theory and practice in elementary probability theory.

According to this perspective one has to use the relation between the ideal experiment and probability theory as a layer for contrasting it with the relation between the concrete experiment and probability theory. From this conception, a large difference between empirical outcomes

"ideal chance experiment —— probability theory"

and theory which differ in comparison to the random deviations expected

"concrete chance experiment —— probability theory"

on the other level, could be seen as indications that there something important has happened in the concrete experiment.

The comparison of the layer with the layer is an indirect application of ideal patterns and expected conditions with the other relation and thus searching for similarities and differences which indicate the specialties of the concrete chance situation in question.

According to this refined perspective on the relation between theory and practice in elementary probability, the two arguments about the existence and the absence of chance could be modified in the following sense:

- If there is a big difference between theoretical expectations and empirical outcomes, then there is a statistical suspicion that "pure" chance conditions (equal probability and independence) are violated; there seems to be some deterministic factor influencing pure chance and thus displaying something specific about the concrete situation to be taken into consideration. The first game indeed can be interpreted as a violation of the condition of stochastic independence.

- The more the number of trials is increased, the more and larger deviations shall occur between the concrete experiment and the theoretical prediction. Because one is playing with concrete die or with some other concrete random generator, the concrete particularities of this device will become the more visible, the more trials are performed; the differences between the idealized idea and the concrete features of the device getting more and more obvious. In the second game, there is perhaps a first hint at a specific factor influencing
ment "Cheating!" of a student in the first episode, as well as the remark
"It is chance if chance is totally eliminated" of another student in the
second episode are reasonable and are based on an appropriate com-
prehension of the complex stochastic game situation. Here some good
students' understandings show up which are open-ended, critical and
useful for developing a theoretical notion of chance.

But according to his own methodical objectives, the teacher takes the
informal aspects of chance and he reduces this concept to a methodical
verbal pattern of justifying the deviations in stochastic experiences. "The
methodical universalization of chance as the counterpart of causal laws
leads to the disregard of the difference between experimental situation
and stochastic model. However large the differences observed between
theoretical predictions and empirical outcomes may be, the methodical
universalization of chance will always be apt to explain this contradic-
tion as something quite natural." (Steinbring 1991a, 518).

4. CONCLUSIONS

An important conclusion from our analysis is that the introduction
of the concept of chance and randomness in the early course of stochastic
requires from the very beginning to respect the differentiation between
a seemingly natural context for this concept and a first theoretical, or
hypothetical idea of this concept. Chance and randomness cannot be
"formally" deduced from statistical experiments, as teachers often intend,
but this at once has to be contrasted with theoretical assumptions con-
cerning idealised models and with probabilistic notions as for instance
the concept of stochastic independence.

The differentiation of the relation between theory and practice in
probability as elaborated here opens an alternative for developing the
formal concept of chance by starting with its informal aspects. In this
way, chance is not reduced by a false kind of abstraction, but despite
its specialization it allows for a mathematical insight into more complex
situations and their preconditions. But now there is no longer a simple
and direct way of applying chance to reality, the application can only
be performed by taking the "whole" theory and by making indirect com-
parisons. This indirect manner of application is due to the theoretical
nature of chance, now in the form of mathematical randomness opera-
tively expressed in the implicit (or axiomatic) formula of multiplying
probabilities. This idea would be a basis for the evolution of the theore-
etical concept of chance, beginning with the informal notion.

Our modified connection between theory and practice in probability
sheds new light on the relation between the situated and the formal
aspects of mathematical knowledge and its understanding. (cf. Lave 1988). The situated context is not simply a motive for the new knowledge, a concrete situation to be abstracted from, but this real context has to be maintained in the further knowledge development. The most important change then will be not to make direct applications or one-sided abstractions, but to explore structures and relationships inherent in the concrete situation and to relate this system of relations as a whole to a theoretical model which then allows for indirect conclusions about the concrete situation.

References:


Dr. Heinz Steinbring, IDM / University of Bielefeld, Postfach 10 01 31, D–33501 Bielefeld, Germany