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Probability and Statistical Mechanics: An Historico-Epistemological Case Study

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PROBABILITY AND STATISTICAL MECHANICS

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I. Introduction

Being studied in this paper are some aspects of the reciproc al relationship between physics and mathematics taking as an example the way the theory of probability developed in the second half of the 19th century. With the aid of a case study special reference is made to the significant role played by the complicated scope of application of the kinetic gas theory for modifying the concept of probability. Demonstrated, above all, is that the theory of probability can no longer be conceived simply as an auxiliary method for physics on the backcloth of principally mechanico-deterministic domains, but that in the discussion about the problems of physics it is becoming a fundamentally theoretical concept and definitive framework for physical reality.

This example clearly shows the necessity for objective reference for mathematical concepts - mathematical concepts cannot "exist" solely as methods detached from any objective validity. Mathematical concepts are characterized rather by just the complementarity principle of modelling and objective-intuitive elements on the one hand and operative-instrumental aspects on the other.

¹) I should like to express my gratitude to R. Biehler, N.H. Jahnke and M. Otte for their critical comments and suggestions.
(cf. Keitel/Otte, 1977 and Otte, 1974) This means to say that the conceptual substance cannot be simply reduced to one of these two moments; in order to characterize the mathematical concept appropriately, their reciprocal relationship must, in particular, be taken into consideration. Falling in line with this the process of evolving and adopting mathematical concepts is determined by the way these complementary aspects act on one another and, thus, can only be comprehended in connection with the formation of the objects specific to them.

The formation of complementary aspects in the concept of probability we want to be understood on the backcloth of the conception of theory evolvement developed by Sneed (cf. Sneed, 1971). In brief in Sneed's conception special importance is attached, above all, to the three following structural characteristics: The core $K$, the domain of intended applications $I$ and the constraints $C$. The particular theory is now conceived as a pair, consisting of the structural core and scope of intended applications $\langle K, I \rangle$, the core encompassing the mathematical apparatus, the domain of intended applications the particular application objects concerned of the theory. This "dual" conceptualization of theory means, inter alia, that a theory cannot be interpreted simply as a system of statements, but one must always take the objective reference of theoretical statements into consideration, the theory being described both by its syntax as well as by its semantic aspects. To a certain extent the function of "mediation" between the structural core and domain of intended applications is attributed to the constraints, which contain in plain, simple terms both a specific "view" of the application objects corresponding to the theory as well as formally syntax relations, making the reference to the structural core. The constraints enable various applications to be recognized of the same type and the theory to be
translated to other application objects for successful application.

The principal problem Sneed intends to treat with his conception is that of "theoretical terms". "Theoretical terms are those concepts of a theory that cannot be defined as a function of observables and yet fulfill an important definitive function within the theory ... It is demonstrated that the relationship of theory and empiricism, of theoretical terms to non-theoretical can only be comprehended appropriately on the basis of dynamic theory conception" (Jahnke, 1978, p. 277).

Our problem of evolving the mathematical concept and here concretely that of probability conforms to the "Problem of theoretical terms" presented here. The development of the theory and the theoretical concept is spurred on by the old domain of intended applications being extended; ultimately the new application objects necessitate the particular concept concerned being modified and further developed.

Now taking the "Gas system" application objects that is new for the concept of probability as a basis, we shall analyze in particular the interconnected conceptional modification of this term. While there used to be initially a purely instrumental conception of probability in this connection, it changes with the formation of the specific characteristics of the application object that comes into question and turns into a theoretically complementary conception. In other words, forming part of the concept of probability is the independent and fundamental significance for the description of the observed physical application object.

Altogether this means amongst other things that the rela-
tion of mathematics to physics cannot be interpreted as that of a range of instruments for physical objects; contrary to such a concept of mathematical formalisation of physical problems, mathematics means that applications and mathematical theories have an intimate reciprocal relationship, which means to say that new applications also develop mathematics substantially.

II. The role of the probability concept up to the first applications in physics

Of great significance for studying the genesis of theoretical concepts within the scope of scientific history is analyzing the causes and reasons stimulating development and generating changes. This is especially true for such a fundamental mathematical concept as that of probability. Ian Hacking has the following to say about this problem in his book "The Emergence of Probability": "There are two ways in which a science develops: in response to problems which itself creates, and in response to problems that are forced upon it from outside." With reference to the concept of probability he goes on: "Only very recently has probability theory been hardy enough to create its own problems and generate its own programmes of research. This stimulus used to come from other disciplines" (Hacking, 1975, p. 4). In his brief survey about the history of the theory of probability Michel Loève also underscores the long lasting and close nexus between the development of methods for the theory of probability and extra-mathematical problems. "Durant près de trois siècles, la motivation des concepts et des problèmes du Calcul des probabilités vint surtout de l'extérieur ... Ce n'est que récemment, durant le vingtième siècle, que le Calcul des probabilités s'est libéré de son rôle d'instrument, et est devenu un membre de plein droit de la famille mathématique" (Loève, 1978, p. 279).
Taking a closer look at the history of the theory of probability will lead to this assessment being corroborated. Whilst in one first originate phase in the theory of probability from the middle of the 17th up to the beginning of the 18th century the formation of the fundamental concepts and elementary theorems in the theory of probability was greatly oriented on the ideal properties of a game of chance, the initially rather intuitive-heuristic application references are, however, of extraordinary significance for the further development of the theory of probability - by way of example, in the field of annuity and insurance calculations. With his theorem at the end of this first phase Jakob Bernoulli opened up the possibility of applying the "combinatorial" theory of probability to varying kinds of random phenomena on a broad basis. Further records indicating that the impulses for the theory of probability developing further during this period principally originated from extra-mathematical fields are, inter alia in addition to Abraham de Moivre's "Treatise of Annuities of Lives", also Condorcet's efforts with regard to his "Mathématique sociale", in which the theory of probability plays a central role (cf., for example, his "Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité de voix", 1785). In addition, in the course of the 18th century measuring problems, above all in astronomy, necessitated a theory of errors of observation being developed, which, in turn, necessitated, on the one hand, the theory of probability, at the same time, however, also serving for new terms, concepts and methods (cf. in this connection, inter alia, the papers of Daniel Bernoulli, Lagrange, Legendre, Laplace and above all Gauss).

The development of the theory of probability has its first climax - and this both as far as the systematology of its methodically theoretical apparatus as well as the relative self-containment with regard to its function for applica-
tion and its theoretical recognition interpretation are concerned - in the work of Pierre Simon de Laplace. In this period the domain of application of the theory of probability was extended to an enormous extent, which proceeded, however, with the old combinatorial conception of probability being adhered to, becoming continually more inappropriate for the new problems. Laplace's definition of probability, which on the backcloth of his mechanico-deterministic view of life ascribes to probability an exclusively instrumental role, is a pregnant expression of this contradiction. Basically all natural processes occur according to reliable mechanical laws and, according to Laplace, "All events, even those which on account of their insignificance do not seem to follow the great laws of nature, are the result of it just as necessarily as the revolutions of the sun" (Laplace, 1951, p. 3). Mostly, however, only "Laplace's demon" is capable of dealing with the plurality of the parameters that come into question, so that imperfect man does not have any option other than implementing the theory of probability as an auxiliary method to approach unknown, but absolute certainty.

In my opinion this purely instrumental concept of probability becomes particularly intelligible in error calculation: Whilst the methods pertaining to the theory of probability necessarily are to be included in the calculation due to various inaccuracies and sources of error, the results obtained are, however, interpreted exclusively mechanically independent of statistical assessments.

Succeeding Laplace three directions for the further development of the theory of probability can be ascertained in the 19th century. In addition to the work of the Petersburg theory of probability school, where inner-theoretical problems became the main spring of development,
an extensive statisticians' school came into being in Western Europe in connection with various problems, such as those in biology, medicine and sociology.

The application of the theory of probability in kinetic gas theory can be designated as the third direction of development. We shall pursue this here and by means of a case study examine the degree to which this complicated scope of application coming into question here altered the role of the theory of probability.

In the following statement of E. Nagel the at this time prevalent interpretation of the theory of probability becomes clear as well as the subsequently occurring further development of this concept: "Although the importance of the main ideas of the mathematical theory of probability for systematizing measurements was quickly recognized in the sciences, the theory of probability was for a long time usually regarded as simply ancillary to the theoretical disciplines. Thus, it was commonly assumed in physics that its laws are statable in a "deterministic" form, such that the positions and velocities of elementary particles at one time are connected in precise ways with the positions and velocities at any other time. It is today a commonplace, however, that some of the most fruitful applications of the theory of probability occur within the theoretical framework of various sciences...."

And later after a brief assessment of Joule's and Maxwell's work in kinetic gas theory Nagel underscores, above all, the significance of Boltzmann's investigations for the theory of probability: "...perhaps the greatest triumph of probability theory within the framework of nineteenth-century physics was Boltzmann's interpretation of the irreversibility of thermal processes; this he was able to do in terms of the most probable distribution of the energies of the molecules of a gas. In consequence, the second law of
thermodynamics can be formulated as a theorem in probability, and irreversible processes turn out to be statistical phenomena." (Nagel, 1939, p.13/14)

III. Mechanics and thermodynamics

At the beginning of the 19th century mechanics was still dominant in physics. In the following comparison Ludwig Boltzmann plainly describes the significance of the mechanical creed for the host of individual physical fields: "Wenn eine Nation große Erfolge erzielt hat im Vergleich mit den in der Nachbarschaft wohnenden, so pflegt sie eine gewisse Hegemonie über die letzteren zu erlangen, ja sie geht nicht selten daran, sie zu unterjochen und sich dienstbar zu machen. Gerade so ergeht es auch mit den wissenschaftlichen Disziplinen. Die Mechanik erlangte bald die Hegemonie in der gesamten Physik. Zunächst unterwarf sich ihr naturgemäß und widerstehlos die Akustik. ... Dasselbe geschah auch mit der Optik. ... Den Feldzug in das Gebiet der Wärmetheorie eröffnete die Mechanik durch die Vorstellung, daß die Wärme eine Bewegung der kleinsten Teilchen der Körper sei. ... Elektrizität und Magnetismus wurden den mechanischen Gesetzen untergeordnet. ..." ¹

(Boltzmann, 1905,p. 311/312). Subsequently Boltzmann additionally described application experiments in mechanics in "animated nature".

¹) "Should a nation achieve great success in comparison with a neighbour, then she will go in for obtaining a certain hegemony over the latter, even setting about not infrequently subjugating and making her subservient. This is precisely what happens with scientific disciplines. Mechanics soon achieved hegemony in overall physics. To begin with acoustics yielded naturally and without resistance... The same also occurred with optics... Mechanics launched a campaign into the field of thermal theory with the notion that heat is a movement of the smallest particles of a body... Electricity and magnetism were subjected to mechanical laws...".
Mark Kac likewise emphasized the striking significance of mechanics for physics in the 19th century: "Mechanics achieves almost the stature of geometry - a model of rigor, precision, and intellectual purity. Mechanics is almost complete: if we only knew the positions and velocities of all bodies today, we could by solving the equations of motion predict the future course of the world. What a magnificent design for the universe!" (Kac, 1974, p. 433)

What made mechanics at that time so successful and gave it this universality? Leopold Infeld judges the position of mechanics in this way: "Es kann nicht überraschen, daß das 19. Jahrhundert eine mechanische Deutung auf alle Bereiche der Naturscheinungen anzuwenden trachtete. Zu Newtons Zeit war die Mechanik der älteste, vertraute und erfolgreichste Zweig der Wissenschaft. Daher mußte man ein geeignetes mechanisches Bild erfinden, wenn man die Erscheinungen der Wärme, des Lichtes und der bewegten Flüssigkeiten erklären wollte. Das ist die Bedeutung der Behauptung, daß die mechanische Auffassung die Physik regierte. Bis ins 19. Jahrhundert stellte sich niemand vor, daß diese Herrschaft der Mechanik gestürzt werden könnte."1) (Infeld, 1953, p. 19) And Albert Einstein stresses subsequent to his criticism levelled at phenomenological physics, which "utilizes terms that are close to experience as possible, while doing homogeneity of underlying principles", thus being

1) "It cannot be a surprise that the 19th century endeavoured to apply a mechanical interpretation to all fields of natural phenomena. At Newton's time mechanics was the oldest, most familiar and most successful branch of science. This was the reason why a suitable mechanical picture had to be found to account for the phenomena of heat, light and moving liquids. This is the importance of the claim that the mechanical conception governed physics. Right up until the 19th century nobody imagined that this dominion of mechanics could be overthrown."
too empirical: "Es ist nach meiner Ansicht die größte Leistung der Newtonschen Mechanik, daß ihre konsequente Anwendung zur Überwindung dieses (phänomenologischen) Standpunktes führte, und zwar auf dem Gebiet der Wärmeerscheinungen. Dies geschah durch die kinetische Gas- theorie und durch die statistische Mechanik überhaupt." 1)
(Quoted according to Broda, 1955, p. 140)

In this statement of Einstein it becomes exemplarily evident how fecund for the development in physics the endeavours of that time were to account for the plurality of physical phenomena mechanically. This was also particularly true for thermodynamics; initially here the opinion was held that a successful starting point had been established in mechanical elucidation. However, in the course of the discussion about atomistics it was shown that a purely mechanical interpretation of heat is not feasible, which then led to the mechanical theory of life being relativized. Persisting in "old" mechanics resulted in a reductionistic programme from the initially meaningful, mechanical elucidations of thermodynamics.

One wanted to account for the phenomenological-thermodynamic governing principles from the standpoint of mechanics by taking as starting point the notion that the macroscopically determinable parameters of a gas, such as, by way of example, temperature, viscosity or heat, result from the motion of the smallest particles, the atoms and molecules. In this way, for instance, definitive experiments in thermodynamic properties are to be found with Daniel Bernoulli (1738) with the aid of the kinetics of atoms.

1) "In my opinion it is the greatest achievement of Newtonian mechanics that its consistent application led to this (phenomenological) point of view being overcome, this occurring namely in the field of thermal phenomena. This happened through the kinetic gas theory and statistical mechanics altogether."
At the same time with this definitive program of thermodynamics by classical mechanics the necessity for integrating theory of probability methods became apparent. The number of the parameters that have to be taken into consideration, i.e. here perhaps the velocity and location of each and every particle, is so gigantic that according to the Laplace interpretation the probability must necessarily be implemented exclusively in its technically instrumental aspects in order to approach the true, i.e. mechanico-deterministic behavior of the gas system. Even the first attempts to deduce thermodynamic laws from the kinetics of the gas system with the assistance of statistical methods were successful. In this way, for example, A. Krönig was able, in 1856, to deduce the Boyle-Mariott Law from kinetics in the form of \( pv = RT \) (pressures multiplied by volume equals constant multiplied by absolute temperature). He integrated the theory of probability just as an auxiliary method: "... die Bahn eines jeden Gasatoms muß ... eine so unregelmäßige sein, daß sie sich der Berechnung entzieht. Nach den Gesetzen der Wahrscheinlichkeitsrechnung wird man jedoch statt dieser vollkommenen Unregelmäßigkeit eine vollkommene Regelmäßigkeit annehmen dürfen."¹ (Krönig, 1856, p. 316). To be found in the work of Rudolf Clausius (as of 1857) are the first formulations for stipulations for the successful implementation of theory of probability methods, such as, by way of example, a form of the later so-called Stoßzahlansatz ². Moreover, in his formulae in addition to isofrequency considerations Clausius also employs isolated qualitative assumptions of the "relative frequency of various absolute velocities of molecules" (cf. P. and T. Ehrenfest, 1911, p. 13 et sequ.).

¹ "... the path of every gas atom must be so irregular that it eludes calculation. However, according to the laws of the theory of probability instead of this complete irregularity one will have to accept complete regularity."

² In principle the assumption about the number of collisions expresses the statistical independence for two groups of colliding molecules. However, the statistically fundamental character of this assumption was only recognized at the conclusion of a long discussion. To begin with attempts were made to establish it purely mechanically.

Ludwig Boltzmann generalized (1868) this Maxwellian formula to multi-atom gases under the effect of gravitation.

For the significance of the theory of probability in this first phase of the definitive experiments in thermodynamics by mechanics it can be determined that probability considerations here were conducted primarily under the perspective of an auxiliary method that necessarily was to be integrated. Achieved was also what was intended: The results obtained by statistical methods could all be assessed in the old definitive framework; here it was a question of "certain" — not statistical — results. The reduction experiment aimed at seemed successful; and initially there was only one problem, which, however, was not yet assessed correctly and could not yet be dealt with in the appropriate manner: The reduction of the second law of thermodynamics to mechanics. The first to refer to this ambiguity was Maxwell, who built up a demon and with this intellectual experiment intended to prove that the second law did not have any mechanical, but only statistical significance (cf. Maxwell's "Theory of heat", 1870). According to S.G. Brush, he "rediculed the German attempts to find mechanical analogies for the Second Law", for he maintained that it was essentially of a statistical nature (cf. Brush, 1976, p. 279).

Below we shall examine this difficulty in greater detail by looking at Ludwig Boltzmann's work, this ambiguity arising in connection with the reduction experiment of the second law to classical mechanics; here these difficulties become particularly clearly manifest both as far as their mathematically explicit representation as well as the severity, with which the discussion about the significance of the second law was conducted by the various groups, are concerned.
IV. The theory of probability and the role of the second law of thermodynamics

Boltzmann had made it his job - very much in the spirit of early 19th century physics - to supply a mechanical explanation of the second law: "Bereits längst ist die Identität des ersten Hauptsatzes der mechanischen Wärmetheorie mit dem Prinzip der lebendigen Kräfte bekannt; dagegen nimmt der zweite Hauptsatz eine eigentümliche exceptionelle Stellung ein und wird sein Beweis auf hier und da nicht einmal sicherer, keinesfalls aber klar vor Augen liegenden Umwegen geführt. Es soll nun der Zweck dieser Abhandlung sein, einen rein analytischen, vollkommen allgemeinen Beweis des zweiten Hauptsatzes der Wärmetheorie zu liefern, sowie den ihm entsprechenden Satz der Mechanik aufzufinden."¹ (Boltzmann, 1866, p.9). Thus does Boltzmann start his paper "Über die mechanische Bedeutung des zweiten Hauptsatzes der Wärmetheorie" - "Concerning the mechanical signigicance of the second law of thermodynamics" (1866). The second law of thermodynamics reflects the experience that "Heat always passes from a hotter to a colder body and not the reverse", and was first formulated by Rudolf Clausius, supported by the work of Carnot, with the aid of the measure of entropy; expressed plainly this theorem states that entropy can only increase in a closed system.

¹ "The identity of the first law of the mechanical thermal theory with the principle of animate forces has already been known for a long time; on the other hand the second law adopts an exceptionnal and peculiar position, the evidence for it being provided in roundabout ways that are not even reliable, let alone clear. Now the purpose of this paper is to provide a purely analytical, completely general proof of the second law of thermodynamics as well as to locate the theorem of mechanics that corresponds to it."
Referring to the entire universe in comparison with the first law, Clausius postulated the following simple form:

"1. The energy of the world is constant.
2. The entropy of the world tends towards a maximum."
(Clausius, 1865, p. 400).

Apart from Boltzmann, a number of scientists - Rankine and Clausius for example - had attempted a mechanical reduction of this momentous theorem of thermodynamics to mechanics. To be said about all these endeavours, including Boltzmann's first experiment, is that they referred to very simplified models of the kinetics of gas particles and all made naturally some kind of assumptions and considerations about the theory of probability. Hannelore Bernhardt assesses this situation in the following manner: "Es zeigte sich also, daß bereits die ersten, zumindest teilweise erfolgreichen Versuche einer mechanischen Deutung des zweiten Hauptsatzes durch Boltzmann und auch durch Clausius nicht ohne gewisse zusätzliche statistische Annahmen auskommen, die nicht ohne weiteres - wenn überhaupt - aus der klassischen Mechanik der Massenpunktsysteme geschlussfolgert werden können."\(^1\) (Bernhardt, 1966, p. 60).

It becomes exemplarily evident from Boltzmann’s assessment of this "mechanical" proof that the significance of statistics, as touched upon in the above quotation, for deducing the second law was not seen by the scientists of that time: "Man sieht leicht, daß unsere Schlüsse von der Bedeutung der darin vorkommenden Größen in der Wärmelehre vollkommen

\(^1\) "It was, therefore, shown that even the first, at least partly successful attempts to interpret the second law mechanically by Boltzmann and also Clausius could not do without certain additional statistical assumptions, which could not be deduced easily - if indeed at all - from classical mechanics of the mass point system."
unabhängig sind und daher zugleich ein Theorem der reinen Mechanik beweisen, welches dem zweiten Hauptsätze gerade in derselben Weise entspricht, wie das Prinzip der lebendigen Kräfte dem ersteren. ..." (Boltzmann 1866, p. 30)

The next, highly important step in the reduction experiments of the second law was introduced by Boltzmann's paper "Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen" - "Further studies concerning thermal balance amongst gas molecules" (1872). Two points are, above all remarkable in this paper for our problem of the relationship of probability and physical knowledge. Firstly, after he had in the meantime also deduced the generalizations of the Maxwellian velocity distribution, Boltzmann referred to the necessity of including theory of probability methods in the reduction experiment: "Lediglich dem Umstand, daß selbst die regellosesten Vorgänge, wenn sie unter denselben Verhältnissen vor sich gehen, doch jedes Mal dieselben Durchschnittswerte liefern, ist es zuzuschreiben, daß wir auch im Verhalten warmer Körper ganz bestimmte Gesetze wahrnehmen. Denn die Moleküle der Körper sind ja so zahlreich und ihre Bewegungen so rasch, daß uns nie etwas anderes als jene Durchschnittswerte wahrnehmbar wird. ... Die Bestimmung der Durchschnittswerte ist Aufgabe der Wahrscheinlichkeitsrechnung." (Boltzmann, 1872, p. 316/317).

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1) "It is easily seen that our conclusions are completely independent of the entities occurring in thermodynamics, thus proving at the same time a theorem of pure mechanics, which corresponds to the second law precisely in the same manner as the principle of animate forces the first ... ."

2) "That we perceive certain specific laws in the behaviour of warm bodies can only be attributed to the circumstance that even the most irregular occurrences - if under the same conditions - supply the same average values each time. For the molecules of the bodies are so numerous and their movement so rapid that we cannot perceive anything other than these average values ... Determining the average values is the task of the theory of probability."
Secondly, after the general deduction of the second law shown here and proof of the unambiguity of Maxwellian velocity distribution, Boltzmann expressly underscores the mechanical significance of this theorem: "Es ist somit streng bewiesen, daß, wie immer die Verteilung der lebendigen Kraft zu Anfang der Zeit gewesen sein mag, sie sich nach Verlauf einer sehr langen Zeit immer notwendig der von Maxwell gefundenen nähern muß." Und weiter unten fährt er fort: "Es ist also hiermit ein analytischer Beweis des zweiten Hauptsatzes auf einem ganz anderen Wege angebahnt, als derselbe bisher versucht wurde." 1) (Boltzmann, 1872, p. 345).

In this connection it should be pointed out that this analytical proof is largely based on the already mentioned assumption about the number of collisions, a prerequisite it was initially believed could be established purely mechanically, which, however, finally proved to be a statistical assumption.

In this paper probability was apparently included in the calculations as an auxiliary method; the results obtained, however, were interpreted purely mechanically without referring to statistics. Mark Kac makes the observation to this proof of Boltzmann: "Since he seemingly used only the laws of mechanics, he could claim that at last he had derived the second law from mechanics." (Kac, 1974, p. 437). Even if Boltzmann did return to the reduction problem some three years later (1875/76), since to begin with he "seems to have been satisfied with the derivation and discussion of the second law in his papers of 1871

1) "It has, thus, been strictly proven that no matter how the distribution of animate force might have been at the beginning of time it will always necessarily approach that discovered by Maxwell after a great deal of time has elapsed." Later he continues: "This means to say that an analytical proof of the second law has herewith been furnished in quite a different manner than has been attempted hitherto."
and 1872." (M.J. Klein, 1971, p. 70), it was these papers that expressed a fundamental problem increasingly more clearly. The difficulty here refers quite generally to the contradiction of on the one hand mechanical reversibility in the dynamical equations of the particles observed and on the other hand the empirically observed irreversibility of the behaviour of the gas system formulated in the second law.

This fundamental contradiction was particularly clearly expressed by the so-called reversibility paradox of Josef Loschmidt (1876). He was especially interested in avoiding the terrible consequences of the heat death connected with the second law and in a number of papers (1876/77) he attempted to prove that contrary to the general Boltzmannian distribution law deviations in temperature occur with effects of gravitation, so that a difference for producing mechanical work is always available.

In this connection Loschmidt constructed simplified gas models and he noted to one of these models: "...wenn wir ... plötzlich die Geschwindigkeiten aller Atome in entgegengesetzter Richtung annehmen, so würden wir damit am Beginn eines Zustandes stehen, dem ebenfalls der Charakter des stationären zuzukommen scheinen würde. Für geraume Zeit wäre das wohl richtig, aber allmählich würde sich der stationäre Zustand gleichsam deteriorieren, und nach dem Verlaufe der Zeit wären wir wieder unfehlbar bei unserem Anfangszustand angekommen..."\(^1\) (Loschmidt, 1876/1877, p. 137). Nothing expresses in such a critical manner the contradiction between the mechanical inter-

\(^1\) "... if we suddenly accept the velocities of all atoms in the opposite direction, then we would be at the start of a condition, which would likewise appear to belong to the character of the stationary. This would probably be correct for considerable time, but gradually the stationary condition would deteriorate so to speak and after the time \(\tau\) has elapsed we would invariably return to our initial condition ... ."
Interpretation of the movement of the smallest particles and the one-sided behaviour of the entropy of the gas system than this reversibility paradox presented here. Reversing all directions of movement in the purely mechanical interpretation contradicts the absolute validity of the second law. Ludwig Boltzmann immediately recognized the importance of this reversibility paradox, which to a certain extent succeeds Maxwell's and Thompson's objections that were formulated on the one hand with the aid of the demon and then also by a comparable reversibility paradox. All objections had one objective - they were aimed at the absolute validity of the second law and necessitated a statistical interpretation.

In his paper "Bemerkungen über einige Probleme der mechanischen Wärmetheorie" - "Notes about some problems of mechanical thermodynamics" (1877a) Boltzmann goes into Loschmidt's objection in detail and emphasizes that it "seems to be of great significance for correct understanding of the second law." (Boltzmann, 1877a, p. 177). In addition for the first time Boltzmann attempted a statistical interpretation of the second law in a comparison with a lottery game, the gas molecules being defined as elastic spheres. "Ein Beweis, daß nach Verlauf einer gewissen Zeit $t_1$ die Mischung der Kugeln mit absoluter Notwendigkeit eine gleichförmige sein müsse, wie immer die Zustandsverteilung zu Anfang der Zeit gewesen sein mag, kann nicht geliefert werden. Dies lehrt schon die Wahrscheinlichkeitsrechnung selbst; denn jede noch so ungleichförmige Zustandsverteilung ist, wenn auch in höchstem Grade unwahrscheinlich, doch nicht absolut unmöglich. Ja, es ist klar, daß jede einzelne gleichförmige Zustandsverteilung, welche über einen bestimmten Anfangszustand nach Verlauf einer bestimmten Zeit entsteht, gerade so unwahrscheinlich ist wie eine einzelne noch so ungleichförmige Zustandsverteilung, gerade so wie im Lottospiel jede einzelne Quinterne ebenso unwahrscheinlich ist wie die Quinterne 1, 2, 3, 4, 5. Nur daher, daß es vielmehr gleichförmige als ungleichförmige Zustandsverteilungen
gibt, stammt die größere Wahrscheinlichkeit, daß die Zustandsverteilung mit der Zeit gleichförmig wird." 1) (Boltzmann, 1877 a, p. 120).

Boltzmann's answer to Loschmidt's objection, thus, runs that "endlessly more initial states" lead rather to a regular rather than to an irregular state distribution; it is, therefore, highly more probable that entropy of the system increases rather than decreases. "Only" statistical significance applies to the second law.

Boltzmann expresses this thus: "Es scheint mir ... der Loschmidt'sche Einwand von großer Wichtigkeit zu sein, weil er zeigt, wie innig der zweite Hauptsatz mit der Wahrscheinlichkeitsrechnung in Verbindung steht, während der erste von ihr ganz unabhängig zu sein scheint." 2) (Boltzmann, 1877a, p. 121)

The possibility of a pure theory of probability calculation of the mechanical equivalent of heat (this possibility being implicitly indicated in this paper) was given by Boltzmann in 1877b in the paper "Über die Beziehung zwischen dem zweiten Hauptsatz der mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung respektive

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1) "A proof that after a certain time \( t_1 \) has elapsed shuffling the spheres must be uniform as an absolute necessity - no matter how the state distribution may have been at the beginning of the time - cannot be provided. This is borne out by the theory of probability itself, since any - no matter how irregular - state distribution is - even if to a very great degree improbable - not absolutely impossible. Yes, it is evident that any single regular state distribution, coming into being after a certain time has elapsed via a certain initial state, is just as improbable as any single - no matter how irregular - state distribution just as in a lottery game any single quintern is just as improbable as the quinterns 1, 2, 3, 4, 5. Only stemming from the fact that there are more regular than irregular state distributions is the greater probability that the state distribution will become regular with time."

2) "It seems to me ... the Loschmidtian objection is of great importance, because he shows how intimately the second law is connected up to the theory of probability, while the first appears to be completely independent of it."
In comparison to the earlier papers the following difference is remarkable here: While earlier on the individual geometrical and impact kinetics of the single particle observed were studied above all and these properties were expressed statistically by average values, the gas system is now characterized entirely statistically by a distribution function. This conceptual transition from the mechanics of single particles to the statistics of the entire system is also underscored by M. J. Klein as being decisive in Boltzmann’s physical thinking and his papers: "In his earlier studies he (Boltzmann) had always been concerned with the statistics of the molecule. The question has always been what is the probability that a molecule has such and such a property, with the answer determined by the molecular distribution function $f$. In his new analysis Boltzmann was concerned with the statistics of the gas as a whole. The question was now what is the probability that the gas is in a state characterized by a certain distribution, and the answer was determined by the permutability measure $P$. This fundamental change in approach must have puzzled Boltzmann's contemporaries." (Klein 1973, Pp. 83/84).

As was indicated at the conclusion of this quotation, it took some time before the statistical interpretations of the second law could be fully accepted by all physicists. The break instigated by Boltzmann was too revolutionary; the fundamental significance of the mechanical view of life was preponderant, a statistical view seeming inconceivable. How revolutionary this shift was is most distinctly seen, in my opinion, by Boltzmann's attitude itself. While it is true that he had provided the pure theory of probability deduction as early as 1877, he was still continuing to look for mechanical argument proofs in the course of the following years. To be mentioned
in this connection are, above all, the ergodic theory initiated by Maxwell and Boltzmann, in which the gas system and its behaviour is transformed into the motion of a point in a high-dimensional space. Here the entire ambiguity is apparently managed in a reliable mechanical framework; however, the impossibility of purely mechanical reduction of the thermodynamic domains soon became obvious. After years of arguments and discussions—particularly with interested physicists in England—Boltzmann noted in 1895 (characteristically enough with reference to his paper from 1877) that the second law is based on the theory of probability and can never be proven mathematically alone from dynamics (cf. in this connection Brush, 1976, p. 622, and Boltzmann, 1985, p. 545 et sequ.).

The fundamentally statistical significance of the assumption about the number of collisions was worked out all the more clearly in the course of these discussions as an expression of statistical independence. It was recognized what importance the theory of probability had had in the old "mechanical" deductions of the second law via the supposedly mechanical "Stoßzahlansatz" (cf. in this connection P. and T. Ehrenfest, 1911, p. 48 et sequ.).

V. Probability and atomistics

As a consistent atomist Boltzmann in the end underscored the significance of the theory of probability unequivocally in the discussion about the mechanical or statistical evaluation of the second law. Taking as his starting point the belief in the existence of atoms he relativized the mechanical validity of the second law in connection with the Loschmidtian reversibility paradox by giving it only "statistical certainty", thus attempting to give an explanation for the fundamental contradiction between microscopic reversibility and macroscopic irreversibility. Not all of Boltzmann's contemporaries drew the same conclusion from this ambiguity. In addition it must
be borne in mind that up until then no atom had been "seen". Indeed atomistics gradually turned into the actual conflict of the discussion. Mark Kac describes the background and causes of the controversy in the following manner: "The price for the atomistic view was, thus, a rejection of determinism and an acceptance of a statistical approach, and to the scientific establishment of the nineteenth century it was too high a price to pay. It was too high especially since no one has ever "seen" atoms or molecules and there were no known violations of the second law." (Kac 1974, p. 439/440).

In addition to Mach, Ostwald and in part also Planck, Poincaré and above all Zermelo were the best known opponents to the atomistics Boltzmann was so vehemently supporting.

Poincaré had proven his recurrence theorem in 1890, which referring to a closed mechanical system with infinitely many particles states that this system behaves quasi-periodically. This means to say that the system returns to its starting point in finite, even if possible very long time with practically certain probability. This objection was, in turn, directed at the absolute validity of the second law; it is interesting to note here that the recurrence theorem thus formulated is itself of a statistical nature. If Boltzmann drew the consequence of a statistical interpretation in the event of other objections, Poincaré and Zermelo were raising their objection precisely to atomistics. H.J. Treder recapitulates this in the following way: "Poincaré und Zermelo ..."
Atomen." 1) (Treder 1971, p. 26/27)
This illustrates the heart of the argument - whilst the atom had been introduced in ancient times and for a long time had been used for explaining heat phenomena, as is the case, for example, with Daniel Bernoulli, it must be taken into further consideration that "atomistics had gone all the way in chemistry since Dalton", however, "only a section of the physicists made atomic science their own." (cf. Broda, 1955, p. 36). A fundamental factor here was that basically the atom was not in its own a physical research object; it served exclusively as a model for elucidation, which otherwise was not attributed with having any reality. And the statistical-mechanical explanations for the thermodynamical phenomena given within the framework of atomic science were extremely complicated, this being the opinion of many physicists. "Why exchange classical simplicity and elegance of thermodynamics for a discipline full of difficulties, uncertainties and paradoxes?" (Kac 1974, p. 440)

In the controversy with Boltzmann Zermelo argued accordingly by adopting the traditional, irrefutable viewpoint of the mechanical view of the world, trying to show that atomistics and thus statistical description was incompatible with this viewpoint; on the other hand, as a convinced atomist, Boltzmann took as his starting point the statistical view from a fundamental position, emphasizing precisely the possibility of the relationship between

1) "Poincaré and Zermelo came to exactly opposing conclusions in the final result for the perspectives of physics... They said: Since atomistics and statistical mechanics based on it - which meant for them materialistic natural elucidation - lead to the second law of thermodynamics not being able to possess any exact validity, atomistics (and, thus, materialistic natural elucidation) was fundamentally false or insufficient. This was the reason behind Zermelo arguing against atomistics in heated polemics with Boltzmann and finally against the very existence of atoms."
mechanical reversibility and statistical irreversibility due to this theory of probability viewpoint. Indeed the essence of the opposing arguments could be expressed as follows: Zermelo took up the old mechanical viewpoint and deduced the unreliability of atomistics from the non-reduceability of the (statistical) interpretation of the second law to mechanics. Boltzmann, on the other hand, deliberately adopted the new viewpoint of statistics and atomistics, thus making the explanation of mechanical aspects to statistical mechanics possible in the reverse manner. New knowledge cannot be completely deduced from old; new knowledge can only account for the old in reverse.

However, before it could finally come to this, the atom had to be accepted as an independent and real research object in overall physics.

An important contribution to this was the mathematical analysis of the Brownian motion by A. Einstein and M. von Smoluchowski. Not only that through the observable motion of a particle suspended in a fluid the impacts of the invisible atoms made atomistics more "plausible"; prerequisites could be made with the aid of the theory constituted, which could then be verified subsequently by experiment. Yet what was probably the most important factor for the argument about atomistics was the experimental proof that had now become feasible to the effect that "there are violations to the second law of thermodynamics". This meant to say that the second law only possesses in fact statistical significance; the theory set up by Boltzmann in formulae does not, therefore, represent "complicating" the basically simple thermodynamic domains, statistics becoming a fundamental description of real physical phenomena.

Khinchin judges the described intimate relation between atomistics and statistics in the following way: "The spe-
cific character of the systems studied in statistical mechanics consists mainly in the enormous number of degrees of freedom which these systems possess. Methodologically, this means that the standpoint to statistical mechanics is determined not by the mechanical nature, but by the particle structure of matter. It almost seems as if the purpose of statistical mechanics is to observe how far reaching are the deductions made on the basis of the atomic structure of matter, irrespective of the nature of these atoms and the laws of their interaction." (Khinchin 1949, p. 9)

Finally it should be briefly mentioned that atomic physics developed at the turn of the century, where the atom and its structure became a research object, led with the Heisenberg uncertainty principle to basic acceptance of the theory of probability: The "mechanical" entities 'impulse' and 'location' become random entities, which limit each other reciprocally and can only be characterized statistically.

VI. Final remarks about the development of the probability concept

Here we have only been able to pursue the most important aspects in the discussion about the significance of the second law of thermodynamics and in addition have limited ourselves to Boltzmann's papers. In this investigation special interest has also been paid to the role of the probability concept that changed in the course of the discussion.

To begin with the theory of probability was integrated into the new-type and complicated ambiguity of the mechanical elucidation of thermodynamics quite in the Laplace sense of an auxiliary method for the incomplete knowledge subject. The reduction problem of the second law to mechanics that was posed in this connection and the objection
formulated against it drew attention in a pregnant manner to a fundamental characteristic of the gas systems that had been understood up till then purely mechanically. With the application objects coming into question here - the systems of many particles, the qualitative differentiation of the micro and macro level played a highly significant role, as was demonstrated in an exemplary manner with the difficulties of the second law in the discrepancy of mechanical reversibility and thermodynamic irreversibility. Basically a new type and specific object was worked out here for the probability concept for the first time in the sense that this concept enabled to handle this contradiction in a fundamental and non-reducible manner while constituting a "probabilistic relation" between the two hierarchical levels of the system. With this change of the object the probability concept has changed from a reducible instrument to a non-reducible theoretical basic concept.

seits die Forschungsrichtung, die von den Systemeigenschaften zu den Elementeigenschaften kommt, und das ist andererseits die Forschungsrichtung, die umgekehrt von den Elementeigenschaften zu den Systemeigenschaften kommt." 1) (Sačkov, 1978, p. 148)

Relative to the framework of theory development outlined at the outset it has become abundantly evident throughout just to what degree the probability concept has completely changed and further developed, starting with the formation of a new-type of application. What can one now learn for physics and mathematics, for the relationship of both to each other and also for the problem of evolving the nomenclature for probability from analyzing this evolvement process?

These questions will only be answered here briefly and in the form of theses, since they necessitate additional argument proofs extending beyond the scope of this paper.

1) "Historically creating the fundamentals of thermodynamics, on the one hand, was prior to the statistical gas theory being worked out and working out the theory of mechanical motion of simple objects, on the other hand, anteceded classical mechanics. In the macroscopic thermodynamic theory of gases the development of atomism raised the question for a specific consolidation of macroscopic laws with the laws of mechanical motion of the individual atoms. In order to realize such a synthesis the laws of the theory of probability were required. In this way the concept of probability became that programme that for the first time on a strictly mathematical basis permitted two fundamental and independent directions of research in system investigation to be combined. This is, on the one hand, the direction of research coming from system properties to element properties and, on the other hand, the direction of research coming vice versa from the element properties to the system properties."
The differentiation between statistical and mechanical domains worked out in this problem relation is of prime significance for physics instruction, this being exemplified concretely by the development of the second law from an initially supposedly mechanical law to a statistical one. The concrete object of the gas system treated here proves to be in a certain manner of general significance for mathematics instruction in the form of a system with varying hierarchical levels. The concept of probability "meets" here with a new type of application object; only through it can the opposite aspects of the system levels be settled and processed homogeneously. It can be said that for the first time one has to do with an object specific to the probability concept inasmuch as this concept can no longer be reduced simply to mechanical or even combinatorial aspects.

From this it soon becomes evident that one has to proceed from a coequal relationship between mathematics and physics - mathematics simply cannot be interpreted as a technical instrumentation for physics.

Proving a momentous result of the way the theory of probability has been evolved in this case study is the significance of probability distribution as a mathematical concept for describing the system appropriately in its various aspects. As a special type of mathematical function probability distribution treats the part/whole relationship in a specific manner. Accordingly the development of the probability concept can be understood and analyzed within the way the function concept has evolved, which is of such high significance for mathematics.
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